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Integer partitions, probabilities and quantum modular forms

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Abstract

What is the probability that the smallest part of a random integer partition of N is odd? What is the expected value of the smallest part of a random integer partition of N ? It is straightforward to see that the answers to these questions are both 1, to leading order. This paper shows that the precise asymptotic expansion of each answer is dictated by special values of an arithmetic L -function. Alternatively, the asymptotics are dictated by the asymptotic expansions of quantum modular forms. A quantum modular form is a function on the rational numbers which has pseudo-modular properties and nice asymptotic expansions near each root of unity. This paper contains five examples involving some of the most famous quantum modular forms of Don Zagier. Additionally, this paper contains new generating function identities for the partition questions relevant to this work and three general circle method asymptotics which may be of independent interest.

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1 Background

Let $\mathcal{P}(N)$ denote the set of all integer partitions of N and write $p(N) = |\mathcal{P}(N)|$ for the number of integer partitions of N . The study of the analytic approximation of partition statistics started with Hardy–Ramanujan asymptotic formula [49], further strengthened by Rademacher’s exact formula [72]. These formulas give for any positive integer R the asymptotic

$$p(N) = \frac{\exp(K\sqrt{N})}{4\sqrt{3}N} \left(\sum_{r=0}^{R-1} p_r N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right) \right),$$

where $K = \frac{2\pi}{\sqrt{6}}$ and

$$p_r = (-2K)^r \sum_{s=0}^r \left(\frac{-K^2}{24} \right)^s \frac{\left(s + \frac{3}{2}, r - s\right)}{s!},$$

$$(v, m) = \frac{\Gamma\left(v + m + \frac{1}{2}\right)}{m! \Gamma\left(v - m + \frac{1}{2}\right)} = \frac{(4v^2 - 1)(4v^2 - 3^2) \cdots (4v^2 - (2m - 1)^2)}{m! 4^m}$$

for $v \in \mathbb{C}$, $m \in \mathbb{Z}_{\geq 0}$. For instance, $p_0 = 1$ and $p_1 = -\frac{1}{K} + \frac{K}{48}$.

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For future reference, set

$$\begin{aligned}
 \mathbf{p}_0 &= 1, \\
 \mathbf{p}_s &= \sum_{j \geq 1} (-1)^j \sum_{r_1 + \dots + r_j = s, r_i \geq 1} p_{r_1} \cdots p_{r_j}.
 \end{aligned}$$

It is natural to pose questions about asymptotics of the number of partitions with prescribed properties. An interesting one is:

What is the probability that the smallest part of an integer partition is odd?

The paper establishes a precise asymptotic answer to this question and a collection of similar ones.

Theorem 1.1 (probability that the smallest part of an unrestricted partition is odd) *Let $sm(\lambda)$ be the smallest part of the partition λ and set*

$$p_{lo}(N) = |\{\lambda \in \mathcal{P}(N) : sm(\lambda) \equiv 1 \pmod{2}\}|.$$

For any positive integer R , one has

$$p_{lo}(N) = \frac{\exp(K\sqrt{N})}{4 \cdot 3^{\frac{1}{2}} N} \left(\sum_{r=0}^{R-1} A_r N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right) \right)$$

as $N \rightarrow \infty$, where

$$A_r = \frac{1}{2(-2K)^r} \sum_{s=0}^r \frac{\sigma_s(s + \frac{3}{2}, r-s)}{s!} \left(\frac{-K^2}{24} \right)^s$$

where σ_n is given by (1.2). So

$$P_N(sm(\lambda) \equiv 1 \pmod{2}) = \frac{p_{lo}(N)}{p(N)} = 1 + \sum_{r=1}^{R-1} \sum_{s=0}^r A_s p_{r-s} N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right),$$

where P_N is the uniform measure on $\mathcal{P}(N)$.

The numbers σ_n appearing in the above theorem are special values of a Hecke L -function. In particular, they arise from the asymptotic expansion of a quantum modular form. Moreover, we show that the asymptotic expansions of many of the most famous quantum modular forms are intimately related to the asymptotics of interesting partition functions. Precisely, two of Zagier’s original four examples [83] and an example of Bettin and Conrey [13] arise in the six main theorems of the present work.

Section 1.2 introduces the quantum modular forms which arise in the present work. In the same spirit as Theorem 1.1, Sect. 1.3 contains four theorems, answering the questions:

- (1) *What is the expected value of the smallest part of an integer partition of N ?*
- (2) *What is the expected value of the smallest part of an integer partition of N with distinct part sizes?*
- (3) *What is the expected value of the largest part of an integer partition of N ?*
- (4) *What is the expected value of the largest part of an integer partition of N with distinct part sizes?*

Section 1.4 contains identities for the generating functions of the relevant counting functions. These identities involve the quantum modular forms introduced in Sect. 1.2. Section 1.5 contains three general circle method propositions which are used to derive the asymptotic expansions used to answer these questions. These are generalizations of Wright’s classic circle method [81] and may be of independent interest.

1.1 A history of partition statistics

There is a long history of the study of partition statistics and questions like those considered in this paper. Perhaps the first systematic study of statistical questions on integer partitions goes back to Erdős and his coauthors, especially Lehner [34] and Turán [35–41]. Additional results on the limiting distribution of partition statistics were found by Fristedt [44] and Pittel [70,71]. While those results, and the results of the present paper, deal with the uniform measure on partitions of n , there has also been substantial interest in random partitions with respect to the Plancherel measure. See the works of Vershik and Kerov [76,77], as well as the work of Oukonkov [68] and the references therein. The work of Erdős, Lehner, Pittel, Turán, Vershik, Kerov, and Oukonkov is largely motivated by the connection between representations of the symmetric groups and integer partitions, see, for instance, the book of MacDonal [63].

There have been many recent works concerned with the rank and crank statistics of integer partitions. These statistics were found by Dyson [32] and Andrews and Garvan [9,45,46] and are motivated by Ramanujan’s congruences for the partition function $p(N)$. Similar to the present work, these statistics have gained a lot of attention because of their connections to exotic modular objects, such as quasimodular forms and mock modular forms [23,24,67,73]. Significant results about the distribution of these statistics appear notably in the works of Bringmann and Mahlburg and their coauthors [15,21,30,31,69].

There remain many more works dealing with statistics of integer partitions and other probabilistic questions about integer partitions. Notable in those are the works dealing with runs and gaps in parts making up a partition [17,18,20,22,48,51,57,78,79].

This paper demonstrates how quantum modular forms and their asymptotic expansions, which involve values of modular L -functions, arise in the resolution of natural integer partition probability problems.

1.2 Quantum modular forms

Zagier [83] introduced the notion of a quantum modular form based on examples from number theory, combinatorics, and quantum invariants of 3-manifolds. Let $k \in \frac{1}{2}\mathbb{Z}$, Γ be a finite index subgroup of $SL_2(\mathbb{Z})$, and $\chi : \Gamma \rightarrow \mathbb{C}^\times$ be a finite-order character. A *quantum modular form* of weight k for Γ is a function $f : \mathbb{P}^1(\mathbb{Q}) \setminus S \rightarrow \mathbb{C}$, where S is a finite subset of $\mathbb{P}^1(\mathbb{Q})$, such that, for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the *period function*

$$h_\gamma(x) := \chi(\gamma)f(x) - (cx + d)^{-k}f\left(\frac{ax + b}{cx + d}\right)$$

extends to a function which is C^∞ or real analytic at all but a finite set of points on $\mathbb{P}^1(\mathbb{R})$. Furthermore, one usually finds an analytic function $f_{\mathbb{C}}(z)$ on $\mathbb{C} \setminus \mathbb{R}$ whose limit as z approaches $x \in \mathbb{Q} \setminus S$ coincides with f .

Many of the examples of quantum modular forms have representations as q -hypergeometric series. Two such series are

$$\begin{aligned} \sigma(q) &= 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (1-q)(1-q^2) \cdots (1-q^n) \\ &= 1 + q - q^2 + 2q^3 - 2q^4 + q^5 + q^7 - 2q^8 + \cdots, \\ \sigma^*(q) &= -2 \sum_{n=0}^{\infty} q^{n+1} (1-q^2)(1-q^4) \cdots (1-q^{2n}) \\ &= -2q - 2q^2 - 2q^3 + 2q^7 + 2q^8 + 2q^{10} + \cdots. \end{aligned}$$

With $q = e^{2\pi iz} = e^{-t}$, define

$$f(z) = \begin{cases} e^{\frac{2\pi iz}{24}} \sigma(e^{2\pi iz}) = q^{\frac{1}{24}} \sigma(q) & z \in \mathbb{H} \cup \mathbb{Q} \\ -e^{-\frac{2\pi iz}{24}} \sigma(e^{-2\pi iz}) = -q^{\frac{1}{24}} \sigma^*(q^{-1}) & z \in \mathbb{H}^- \cup \mathbb{Q}. \end{cases}$$

Using the results of Andrews et al. [7], Cohen [29], and Lewis and Zagier [60], Zagier [83] showed that

$$h(z) := (2z + 1)f\left(\frac{z}{2z + 1}\right) - e^{\frac{2\pi i}{24}} f(z)$$

is a function which is holomorphic on all of $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Moreover, the function $t \mapsto f\left(\frac{it}{2\pi}\right)$ has an asymptotic expansion near every value $t \in \mathbb{Q}$. For instance, near $t = 0$

$$\begin{aligned} e^{-\frac{t}{24}} \sigma(e^{-t}) &= -e^{-\frac{t}{24}} \sigma^*(e^t) \sim \sum_{n=0}^{\infty} \frac{\sigma_n}{n!} \left(\frac{t}{24}\right)^n \\ &= 2 - 50 \left(\frac{t}{24}\right) + \frac{5855}{2} \left(\frac{t}{24}\right)^2 \\ &\quad - \frac{1538066}{3!} \left(\frac{t}{24}\right)^3 + \frac{718108610}{4!} \left(\frac{t}{24}\right)^4 - \cdots, \end{aligned}$$

where the σ_n are integers given in terms of a Hecke L -function. To describe σ_n , let $T(n)$ be the number of inequivalent solutions to

$$u^2 - 6v^2 = n \tag{1.1}$$

such that $u + 3v \equiv \pm 1 \pmod{12}$ minus the number of such solutions with $u + 3v \equiv \pm 5 \pmod{12}$. Here two solutions (u, v) and (u', v') to (1.1) are *equivalent* if $(u' + v'\sqrt{6}) = \pm(5 + 2\sqrt{6})^r(u + v\sqrt{6})$ for some integer r ; otherwise they are said to be *inequivalent*. Define the Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} \frac{T(n)}{n^s} \quad (\text{Re}(s) > 1).$$

Then $D(s)$ is a Hecke L -function and hence has analytic continuation to the complex s -plane. Set

$$\sigma_n := (-1)^n D(-n). \tag{1.2}$$

See [16] for more on the values $D(-n)$ including their asymptotic properties.

Thus Theorem 1.1 shows the asymptotic expansion of this quantum modular form arises in the probability theory of partitions.

Our work requires two more quantum modular forms. Zagier [82,83] showed that the Kontsevich’s “strange” function

$$F(q) := \sum_{n=0}^{\infty} (1 - q)(1 - q^2) \cdots (1 - q^n)$$

fits into the framework of a quantum modular form. This q -series is defined only at roots of unity q . Combining the results of [82] with the results of Bryson, Ono, Pitman, and the second author [26] it is possible to construct a function on $(\mathbb{C} \setminus \mathbb{R}) \cup \mathbb{Q}$ analogous to f above. Zagier [82] proved

$$e^{-\frac{t}{24}} F(e^{-t}) \sim \sum_{n=0}^{\infty} \frac{T_n}{n!} \left(\frac{t}{24}\right)^n = 1 + 23 \left(\frac{t}{24}\right) + \frac{1681}{2} \left(\frac{t}{24}\right)^2 + \frac{257,543}{3!} \left(\frac{t}{24}\right)^3 + \cdots$$

The T_n , called *Glaiser’s numbers*, are special values of the Dirichlet L -series $L(\chi_{12}, s)$ where $\chi_{12} = \left(\frac{12}{\cdot}\right)$ is a quadratic character of conductor 12. More precisely

$$T_n = \frac{6\sqrt{3}}{\pi^2} (2n + 1)! \left(\frac{12}{\pi^2}\right)^n L(\chi_{12}, 2n + 2). \tag{1.3}$$

Finally, the third quantum modular form that plays a role in our work is

$$S_0(z) = \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} = \sum_{n=1}^{\infty} d(n)q^n, \quad \text{with } q = e^{2\pi iz}.$$

Bettin and Conrey [13] showed that

$$S_0(z) - \frac{4}{z} S_0\left(-\frac{1}{z}\right) = \frac{1}{4} \left(1 - \frac{1}{z}\right) - \frac{1}{4} \psi_0(z),$$

where ψ_0 is the period function of an Eisenstein series. Moreover, they established the asymptotic

$$S_0(it/2\pi) \sim -\frac{\log(t) + \gamma}{t} + \frac{1}{4} + \sum_{n=1}^{\infty} \frac{B_{2n}^2 t^{2n-1}}{(2n)!(2n)} \quad (t \rightarrow 0),$$

where $\gamma = 0.5772156 \dots$ is Euler’s constant [58], B_n is the n th Bernoulli number, and

$$B_{2n} = \frac{(-1)^{n+1} 2(2n)!}{(2\pi)^{2n}} \zeta(2n),$$

where $\zeta(s)$ is the Riemann zeta function. See Sect. 4.2 for further discussion on $S_0(z)$. In the remainder we abuse notation and write $S_0(q) = S_0(z)$. For future reference we define the coefficients b_n by

$$e^{\frac{t}{24}} \left(\frac{\gamma}{t} - \frac{1}{4} + \sum_{n=1}^{\infty} \frac{B_{2n}^2 t^{2n-1}}{(2n)!(2n)} \right) =: \frac{1}{t} \sum_{n=0}^{\infty} b_n t^n. \tag{1.4}$$

For instance, $b_0 = \gamma$, $b_1 = \frac{\gamma}{24} - \frac{1}{4}$, and $b_2 = \frac{\gamma}{1152} - \frac{1}{288}$.

1.3 Asymptotic theorems

This section contains five theorems, each of which relates the asymptotic expansion of a particular partition probability function to the asymptotic expansion of a quantum modular form.

Let $\mathcal{D}(N)$ denote the set of all integer partitions of N into distinct parts; set $q(N) = |\mathcal{D}(N)|$. It follows from [49] or [50] (see also Proposition 1.8) that for any $R \geq 0$

$$q(N) = \frac{\exp\left(K'\sqrt{N}\right)}{4 \cdot 3^{\frac{1}{4}} N^{\frac{3}{4}}} \left(\sum_{r=0}^{R-1} q_r N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right) \right),$$

where $K' = \frac{\pi}{3^{\frac{1}{2}}}$ and

$$q_r = (-2K')^{-r} \sum_{s=0}^r \left(\frac{K'^2}{24}\right)^s \frac{(s+1, r-s)}{s!}.$$

In particular, $q_0 = 1$ and $q_1 = -\frac{3}{8K'} - \frac{K'}{48}$. For future reference, define

$$\begin{aligned} \mathbf{q}_0 &= 1, \\ \mathbf{q}_s &= \sum_{j \geq 1} (-1)^j \sum_{r_1 + \dots + r_j = s, r_i \geq 1} q_{r_1} \cdots q_{r_j}. \end{aligned}$$

Theorem 1.2 (Expected value of the smallest part in a distinct part partition) *If R is a positive integer, then*

$$e_{sd}(N) := \sum_{\lambda \in \mathcal{D}(N)} \text{sm}(\lambda) = \frac{\exp\left(K'\sqrt{N}\right)}{4 \cdot 3^{\frac{1}{4}} N^{\frac{3}{4}}} \left(\sum_{r=0}^{R-1} B_r N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right) \right)$$

as $N \rightarrow \infty$, where

$$B_r = (-2K')^{-r} \sum_{s=0}^r \left(\frac{K'^2}{24}\right)^s \frac{\sigma_s (s+1, r-s)}{s!}$$

Consequently, the expected value of the smallest part of a partition with respect to the uniform measure on the distinct part partitions of N is

$$E_{d,N}(\text{sm}(\lambda)) = 2 + \sum_{r=1}^{R-1} \sum_{s=0}^r B_s \mathbf{q}_{r-s} N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right).$$

Remark Fristedt [44, Section 9] proved that the probability that j is the smallest part of a random distinct part partition is 2^{-j} . Consequently, the expected value is $\sum_j j 2^{-j} = 2$. Note in our asymptotic that $B_0 = 2$.

The following theorem is offered for contrast to the previous theorem. Let $\ell(\lambda)$ be the number of parts in λ .

Theorem 1.3 *One has*

$$\sum_{\lambda \in \mathcal{D}(N)} (-1)^{\ell(\lambda)+1} \text{sm}(\lambda) = \sum_{d|N} 1.$$

Consequently, as $N \rightarrow \infty$

$$\begin{aligned} E_{d,N}(\text{sm}(\lambda) : \ell(\lambda) \equiv 1 \pmod{2}) \\ = E_{d,N}(\text{sm}(\lambda) : \ell(\lambda) \equiv 0 \pmod{2}) + o\left(N^{\frac{3}{4}+\epsilon} \exp\left(-K'\sqrt{N}\right)\right). \end{aligned}$$

Remark It is a classical fact that $\sum_{d|N} 1 = o(N^\epsilon)$ for any $\epsilon > 0$ (see for instance [64, page 56]).

Theorem 1.4 (Expected value of the smallest part of an unrestricted partition) *If R is a positive integer, then*

$$e_{\text{sm}}(N) := \sum_{\lambda \in \mathcal{P}(N)} \text{sm}(\lambda) = \frac{\exp\left(K\sqrt{N}\right)}{4 \cdot 3^{\frac{1}{2}} N} \left(\sum_{r=0}^{R-1} D_r N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right) \right)$$

as $N \rightarrow \infty$, where

$$D_r = (-2K')^{-r} \sum_{s=0}^r \left(\frac{-K^2}{24}\right)^s \frac{T_s(s+1, r-s)}{s!}$$

and T_s is given by (1.3).

Consequently, the expected value of the largest part of an unrestricted partition of N , with respect to the uniform measure, is

$$E_N(\text{sm}(\lambda)) = 1 + \sum_{r=1}^{R-1} \sum_{s=0}^r D_s p_{r-s} N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right).$$

Remark Again, by [44] a random partition of N almost surely has a part of size 1 and so $E_N(\text{sm}(\lambda)) \sim 1$.

The above theorems concern the smallest part of a partition. We now turn our attention to the largest part of a partition.

Theorem 1.5 (Expected value of the largest part of an unrestricted partition) *As $N \rightarrow \infty$*

$$e_{\text{lg}}(N) := \sum_{\lambda \in \mathcal{P}(N)} \text{lg}(\lambda) = \frac{\exp(K\sqrt{N})}{4\pi \cdot 2^{\frac{1}{2}} N^{\frac{1}{2}}} \left(\log N + 2\gamma - 2 \log \frac{K}{2} + O\left(N^{-\frac{1}{2}} \log N\right) \right),$$

where $\gamma = 0.5772156\dots$ is Euler's constant.

Consequently, the expected value with respect to the uniform measure on the unrestricted partitions of N is

$$E_N(\text{lg}(\lambda)) = \frac{\sqrt{6N}}{\pi} \log\left(\frac{\sqrt{6N}}{\pi}\right) + \frac{\sqrt{6N}}{\pi} \gamma + O(\log(N)).$$

Remark Erdős and Lehner showed in [34] that

$$\lim_{N \rightarrow \infty} \mathbf{P}_N \left(\frac{\pi}{\sqrt{6N}} \text{lg}(\lambda) - \log\left(\frac{\sqrt{6N}}{\pi}\right) < \nu \right) = e^{-e^{-\nu}}.$$

From this it is straightforward to deduce

$$E_N(\text{lg}(\lambda)) \sim \frac{\sqrt{6N}}{\pi} \log\left(\frac{\sqrt{6N}}{\pi}\right) + \frac{\sqrt{6N}}{\pi} \gamma.$$

Curiously, in the above the γ occurs from the asymptotic expansion of the Riemann zeta function $\zeta(s)$ near $s = 1$. It might be interesting to compare with the integral $\int_{-\infty}^{\infty} \nu e^{-e^{-\nu}-\nu} d\nu$ which occurs in the limiting distribution of Erdős and Lehner.

Remark Theorem 1.5, as stated, was previously established by Kessler and Livingston [53] with a slightly worse error term and also by Grabner et al. [48]. The present method makes the structure of the full asymptotic expansion apparent and demonstrates the role of the quantum modular form $S_0(z)$ in determining the expansion. The next theorem has the full asymptotic expansion for the case of partitions with distinct parts.

Theorem 1.6 (Expected value of the largest part of a distinct part partition) *For any $R \geq 0$ as $N \rightarrow \infty$*

$$e_{\text{id}}(N) := \sum_{\lambda \in \mathcal{D}(N)} \text{lg}(\lambda) = \frac{\exp(K' \sqrt{N})}{4\pi \cdot 3^{-\frac{1}{4}} N^{\frac{1}{4}}} \left(\sum_{r=0}^{R-1} (F_r'' + F_r' + F_r \log N) N^{-\frac{r}{2}} + O(N^{-\frac{R}{2}} \log N) \right),$$

where

$$\begin{aligned} F_r'' &= 2(-2K')^{-r} \sum_{s=0}^r (-K'^2)^s (s, r-s) b_s, \\ F_r' &= (-2K')^{-r} \sum_{s=0}^r \left(\frac{-K'^2}{24} \right)^s \frac{(-2) \log(K'/2) (s, r-s)}{s!} \\ &\quad - \sum_{s=0}^r \frac{(8\pi)^{\frac{1}{2}} (K')^{2s-r} a_{r-s-1}^*(s)}{96^s s!}, \\ F_r &= (-2K')^{-r} \sum_{s=0}^r \left(\frac{-K'^2}{24} \right)^s \frac{(s, r-s)}{s!}, \end{aligned}$$

where b_s is defined in (1.4) in terms of Bernoulli numbers and a_s^* is defined in Proposition 1.9 in terms of values of the Gamma function.

1.4 Generating function identities

This section discusses the generating functions for the partition problems above. Euler [6] was the first to show the generating function identities

$$\begin{aligned} P(q) &:= \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{(q; q)_{\infty}} \\ Q(q) &:= \sum_{n=0}^{\infty} q(n)q^n = \prod_{n=1}^{\infty} (1+q^n) = (-q; q)_{\infty} \end{aligned}$$

where $(a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$ and $(a; q)_{\infty} = \prod_{j=0}^{\infty} (1 - aq^j)$.

The following theorem gives results for the generating functions studied in our main theorems.

Theorem 1.7 *In the notation of Theorems 1.1, 1.2, 1.4, 1.5, and 1.6*

$$\sum_{N=0}^{\infty} p_{\text{lo}}(N)q^N = \sum_{\substack{\lambda \in \mathcal{P} \\ \text{sm}(\lambda) \equiv 1 \pmod{2}}} q^{|\lambda|} = P(q) \cdot \sum_{n=0}^{\infty} (-1)^n ((q; q)_n - (q; q)_{\infty}) \tag{1.5}$$

$$= P(q) \cdot \frac{1}{2} \sigma(q) - \frac{1}{2},$$

$$\sum_{N=0}^{\infty} e_{\text{sd}}(N)q^N = \sum_{\lambda \in \mathcal{D}} \text{sm}(\lambda)q^{|\lambda|} = -Q(q) \cdot \sigma^*(q) - S_0(q), \tag{1.6}$$

$$\sum_{N=0}^{\infty} e_{\text{sm}}(N)q^N = \sum_{\lambda \in \mathcal{P}} \text{sm}(\lambda)q^{|\lambda|} = P(q) \cdot \sum_{n=0}^{\infty} ((q; q)_n - (q; q)_{\infty}), \tag{1.7}$$

$$\sum_{N=0}^{\infty} e_{\text{lg}}(N)q^N = \sum_{\lambda \in \mathcal{P}} \text{lg}(\lambda)q^{|\lambda|} = P(q) \cdot \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}, \tag{1.8}$$

$$\sum_{N=0}^{\infty} e_{\text{id}}(N)q^N = \sum_{\lambda \in \mathcal{D}} \text{lg}(\lambda)q^{|\lambda|} = Q(q) \cdot \left(S_0(q) - \frac{1}{2} \right) + \frac{1}{2} \sigma(q), \tag{1.9}$$

where

$$P(q) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)}, \quad Q(q) = \prod_{n=1}^{\infty} (1 + q^n),$$

$$\sigma(q) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} \prod_{j=1}^n (1 - q^j), \quad \sigma^*(q) = -2 \sum_{n=0}^{\infty} q^{n+1} \prod_{j=1}^n (1 - q^{2j}),$$

$$S_0(q) = \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}.$$

Remark At all roots of unity, to infinite order,

$$\sum_{n=1}^{\infty} (-1)^n ((q; q)_n - (q; q)_{\infty}) = \sum_{n=1}^{\infty} (-1)^n (q; q)_n = \frac{1}{2} \sigma(q)$$

and

$$\sum_{n=1}^{\infty} ((q; q)_n - (q; q)_{\infty}) = F(q).$$

Thus, these forms behave as quantum modular forms near roots of unity. These identities are discussed in more detail in the proof of Theorem 1.1 and Sect. 4.4.

Remark The identities of Theorem 1.7 are all of the shape

$$\text{Generating Function} = (P(q) \text{ or } Q(q)) \times \text{Sieving Function} + \text{Error Function}.$$

In the above cases, the sieving functions are quantum modular forms. For example in the first equality the “sieving function” is

$$\sum_{n=1}^{\infty} (-1)^n ((q; q)_n - (q; q)_{\infty}) = q + q^3 - q^4 - q^8 + q^{10} - q^{13} + q^{14} + q^{15} - q^{17} + q^{18} - q^{19} + \dots.$$

Consequentially,

$$p_{10}(n) = p(n - 1) + p(n - 3) - p(n - 4) - p(n - 8) + p(n - 10) + p(n - 13) + \dots$$

This is a typical behavior. See the book of Stanley [75] for discussion of sieving in enumerative combinatorics. See the papers of Andrews [4,5], Bringmann and Mahlburg [22], Kim and Lovejoy [54], Kim and Jo [52], Kim et al. [55,56], and Grabner et al. [48] for more examples of this phenomenon. Most commonly, the “sieving function” is a partial theta function, which is known to be a quantum modular form. See [43,59] and [14] for details about the quantum modularity of partial theta functions. In some other cases, the sieving function may be a mock modular form, which may have exponential singularities at some roots of unity, see [4,12,74] for examples of this sort.

We have chosen the above examples because they have simple partition interpretations and because the quantum modular forms arising are among the most exotic.

Remark Define for a natural number k the q -series $M_k(q) = \sum_{n=1}^{\infty} \frac{n^k q^n}{(q; q)_n}$; here $(q; q)_n := \prod_{j=1}^n (1 - q^j)$. Then the coefficient of q^n in $M_k(q)$ is, up to normalizing by $1/p(n)$, the k th-moment of the largest part of a partition of n , namely $\sum_{\lambda} \lg(\lambda)^k$ where the sum runs over all partitions λ of n . Now define $S_k(q) = \sum_{n=1}^{\infty} \frac{n^k q^n}{1 - q^n}$. One may show that S_k and M_k are related via the recursion

$$M_k(q) = \sum_{j=0}^{k-1} \binom{k-1}{j} S_{k-1-j}(q) M_j(q).$$

If k is an odd integer, then S_k is (essentially) an Eisenstein series. If k is an even integer, then S_k is not an Eisenstein series, but it still has pseudo-modularity (see Bettin and Conrey [13]). Thus, the method of this paper can be applied to compute an asymptotic expansion for each of the moments of the largest parts of partitions.

1.5 Wright’s asymptotic expansion and analogues

In this subsection we state generalizations of Wright’s asymptotic expansion using the circle method. This section contains three propositions in a quite general form with the hope that it might be useful for other researchers. All of the propositions concern the asymptotics for the q -coefficients of functions of the form $L(q)\xi(q)$ where $\xi(q)$ has its “main” exponential singularity as $q \rightarrow 1$ and $L(q)$ has polynomial or logarithmic behavior as $q \rightarrow 1$.

Throughout this section let q denote a complex variable with $|q| < 1$ and $q \notin \mathbb{R}_{\leq 0}$. Put $q = e^{-t}$, so $t \in \mathbb{C}$ satisfies $\text{Re } t > 0$ and $|\text{Im } t| < \pi$.

The setup is as follows. Let $L(q)$ and $\xi(q)$ be two functions analytic for $|q| < 1$ and $q \notin \mathbb{R}_{\leq 0}$ so that $L(q)\xi(q)$ is analytic for $|q| < 1$ and $q \notin \mathbb{R}_{\leq 0}$. We are interested in the asymptotic expansion of

$$V(N) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{L(q)\xi(q)}{q^{N+1}} dq \quad (N \rightarrow \infty), \tag{1.10}$$

where \mathcal{C} is a circle centered at the origin with radius smaller than one and where q describes \mathcal{C} counter-clockwise. Integral (1.10) will arise from the combinatorial problems described in Sect. 1.

Let $0 < \delta < \frac{\pi}{2}$ and $c > 0$ be fixed constants. Set $\eta = \frac{c}{\sqrt{N}}$. Let \mathcal{C} be the circle on the q -plane which is centered at 0 and of radius $e^{-\eta}$. Let \mathcal{C}_1 be the arc of \mathcal{C} such that $q \in \mathcal{C}_1$ precisely when $|\arg t| < \frac{\pi}{2} - \delta$; set $\mathcal{C}_2 = \mathcal{C} - \mathcal{C}_1$. We make available four running assumptions:

HYPO 1 for every positive integer k , as $|t| \rightarrow 0$ in the bounded cone $|\arg t| < \frac{\pi}{2} - \delta$, either

$$L(e^{-t}) = \frac{1}{t^B} \left(\sum_{s=0}^{k-1} \alpha_s t^s + O_\delta(t^k) \right), \tag{1.11}$$

where $\alpha_s \in \mathbb{C}$ and B is a real constant, in which case we say $L(e^{-t})$ has *polynomial type near 1*, or

$$L(e^{-t}) = \frac{\log t}{t^B} \left(\sum_{s=0}^{k-1} \alpha_s t^s + O_\delta(t^k) \right), \tag{1.12}$$

where $\alpha_s \in \mathbb{C}$ and B is a real constant, in which case we say $L(e^{-t})$ has *logarithmic type near 1*;

HYPO 2 as $|t| \rightarrow 0$ in the bounded cone $|\arg t| < \frac{\pi}{2} - \delta$

$$\xi(e^{-t}) = t^\beta e^{c^2/t} (1 + O_\delta(e^{-\gamma/t})), \tag{1.13}$$

where β and γ are real constants such that $\beta \geq 0$ and $\gamma > c^2$;

HYPO 3 as $|t| \rightarrow 0$ in the bounded cone $\frac{\pi}{2} - \delta \leq |\arg t| < \frac{\pi}{2}$

$$L(e^{-t}) \ll_\delta t^{-C}, \tag{1.14}$$

where $C = C(\delta)$ is a positive real constant;

HYPO 4 as $|t| \rightarrow 0$ in the bounded cone $\frac{\pi}{2} - \delta \leq |\arg t| < \frac{\pi}{2}$

$$|\xi(e^{-t})| \ll_\delta \xi(|e^{-t}|) e^{-\frac{\delta'}{t}}, \tag{1.15}$$

where $\delta' = \delta'(\delta)$ is a positive real constant.

Remark The assumptions HYPO 1 and HYPO 2 put asymptotics on L and ξ on the “major arc” \mathcal{C}_1 near 1, whereas the assumptions HYPO 3 and HYPO 4 require that L and ξ are small on the “minor arc” \mathcal{C}_2 away from 1.

The following result is due to Wright [81, Section 6]:

Proposition 1.8 (Asymptotic expansion in the polynomial-type case) *Suppose that the four running assumptions are satisfied. Suppose further that $L(q)$ has polynomial type near 1. Then*

$$V(N) = e^{2c\sqrt{N}} N^{\frac{1}{4}(2B-2\beta-3)} \left(\sum_{r=0}^{R-1} p_r N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right) \right) \quad (N \rightarrow \infty),$$

where

$$p_r = \sum_{s=0}^r \alpha_s w_{s,r-s}$$

with the α_s given by HYPO 1 (1.11) and the $w_{s,r}$ given by

$$w_{s,r} = \frac{\left(-\frac{1}{4c}\right)^r c^{s+\beta-B+\frac{1}{2}}}{2\pi^{\frac{1}{2}}} (s + \beta - B + 1, r). \tag{1.16}$$

The following is an analogue of Proposition 1.8, now for the case of logarithmic type:

Proposition 1.9 (Asymptotic expansion in the logarithmic type with integer-order case) *Suppose that the four running assumptions are satisfied. Suppose further that $L(q)$ has logarithmic type near 1 and that $B - \beta = 1$ where B is the constant in HYPO 1 (1.12) and β is the constant in HYPO 2 (1.13). Set*

$$\begin{aligned} a_r^*(n) &= - \sum_{j=0}^{\min(n-1,r)} \frac{\Gamma(n+1)(-2)^{2j-r}}{\Gamma(n-j)(j+1)(2\pi)^{\frac{1}{2}}} (n-1-j, r-j) && (r, n \in \mathbb{Z}_{\geq 0}), \\ l_{s,r} &= \frac{\left(-\frac{1}{4c}\right)^r c^{s-\frac{1}{2}}}{-4\pi^{\frac{1}{2}}} (s, r) && (s, r \in \mathbb{Z}_{\geq 0}), \\ l'_{s,r} &= \frac{\left(-\frac{1}{4c}\right)^r c^{s-\frac{1}{2}} \log c}{2\pi^{\frac{1}{2}}} (s, r) + \delta_{r \geq 1} \frac{c^{s-r-\frac{1}{2}}}{2^{r+\frac{1}{2}}} a_{r-1}^*(s) && (s, r \in \mathbb{Z}_{\geq 0}). \end{aligned}$$

One has

$$V(N) = e^{2c\sqrt{N}} N^{-\frac{1}{4}} \left(\sum_{r=0}^{R-1} N^{-\frac{r}{2}} (q'_r + q_r \log N) + O\left(N^{-\frac{R}{2}} \log N\right) \right) \quad (N \rightarrow \infty),$$

where

$$q_r = \sum_{s=0}^r \alpha_s l_{s,r-s} \quad \text{and} \quad q'_r = \sum_{s=0}^r \alpha_s l'_{s,r-s}$$

with the α_s given by HYPO 1 (1.12).

Remark We will see later from our proof that one can extend Proposition 1.9 to the case $B - \beta \in \mathbb{Z}$ (of course with more involved asymptotic formulas). Hence the nature of Proposition 1.9 is of logarithmic type with integer order.

The following is another analogue of Proposition 1.8, also for the case of logarithmic type:

Proposition 1.10 (Asymptotic expansion in the logarithmic type with half-integer-order case) *Suppose that the four running assumptions are satisfied. Suppose further that $L(q)$ has logarithmic type near 1 and that $B - \beta = \frac{1}{2}$ where B is the constant in HYPO 1 (1.12) and β is the constant in HYPO 2 (1.13). Then*

$$V(N) = e^{2c\sqrt{N}} N^{-\frac{1}{2}} \frac{-\alpha_0}{4\pi^{\frac{1}{2}}} \left(\log N - 2 \log c + O(N^{-\frac{1}{2}} \log N) \right) \quad (N \rightarrow \infty).$$

Remark One can write down a full asymptotic expansion for $V(N)$ in Proposition 1.10 and can extend to the case $B - \beta \in \frac{1}{2} + \mathbb{Z}$ (of course with more involved asymptotic formulas). Hence the nature of Proposition 1.10 is of logarithmic type with half-integer order.

The main idea of the proofs of Propositions 1.8, 1.9, and 1.10 is the circle method. The main technical ingredients are various asymptotic expansions of Bessel functions. The reader who is willing to trust Propositions 1.8, 1.9, and 1.10 can skip Sect. 3.

2 Proof of generating function identities

This section contains the proof of the generating function identities of Theorem 1.7. The proof uses the following theorems of Andrews and Freistas.

Theorem 2.1 (Theorem 4.1 of [8]) *Let g be a function defined by the series $g(x) = \sum_{n=0}^{\infty} g_n x^n$. Then*

$$\sum_{n=0}^{\infty} g_n \left(\frac{(t)_n}{(a)_n} - \frac{(t)_{\infty}}{(a)_{\infty}} \right) = \frac{(t)_{\infty}}{(a)_{\infty}} \sum_{n=1}^{\infty} \frac{(a/t)_n}{(q)_n} g(q^n) t^n.$$

Theorem 2.2 (Theorem 4.4 of [8]) *One has*

$$\sum_{n=0}^{\infty} \left(\frac{(a)_n (b)_n}{(c)_n (q)_n} - \frac{(a)_{\infty} (b)_{\infty}}{(c)_{\infty} (q)_{\infty}} \right) = \frac{(a)_{\infty} (b)_{\infty}}{(c)_{\infty} (q)_{\infty}} \sum_{n=1}^{\infty} \left(\frac{(q/b)_n b^n}{(q)_n} + \frac{(c/a)_n a^n}{(b)_n} \right) \frac{1}{1 - q^n}.$$

Proof of Theorem 1.7 (1) Recall that $p_{1o}(N)$ is the number of partitions of N for which $sm(\lambda) \equiv 1 \pmod{2}$. We begin by establishing

$$\sum_{N=1}^{\infty} p_{1o}(N) q^N = \sum_{\lambda: sm(\lambda) \equiv 1 \pmod{2}} q^{|\lambda|} = \sum_{n=1}^{\infty} \frac{q^n}{(q)_{n-1} (1 - q^{2n})}. \tag{2.1}$$

Indeed, the first equality is the definition of $p_{1o}(N)$. Let $p'_{1o}(N)$ denote the number of partitions of N in which the largest part, indexed by n say, occurs an odd number of times. Then the third member of (2.1) is precisely $\sum_{N=1}^{\infty} p'_{1o}(N) q^N$.

Recall that we can express a partition λ of N as a decreasing sequence $(\lambda_1, \dots, \lambda_r)$ with $\sum_{i=1}^r \lambda_i = N$ by a *Young diagram*, which by definition is a rectangular array of N boxes, with r rows where the i th-row is of length λ_i . The *conjugate partition* λ' of λ is the partition whose Young diagram is obtained by reflecting the Young diagram of λ about the diagonal so that rows become columns and columns become rows. For example, the conjugate partition of the partition $(4, 3, 2, 2, 1)$ is the partition $(5, 4, 2, 1)$.

Taking the conjugate partitions gives a bijection between the partitions of N in which the largest part occurs an odd number of times with the partitions of N with the smallest part being odd. In particular we have $p_{1o}(N) = p'_{1o}(N)$; this shows the second equality of (2.1). The desired identity is an immediate consequence of the theorem of Andrews and Freistas, Theorem 2.1 (taking $g_n = (-1)^n, a = 0, t = q$).

(2) Recall that $e_{sd}(N)$ is the sum of the smallest parts of the partitions of N with distinct parts. By definition it is clear that

$$\sum_{N=0}^{\infty} e_{sd}(N) q^N = \sum_{\lambda} sm(\lambda) q^{|\lambda|} = \sum_{n=0}^{\infty} n q^n (-q^{n+1}; q)_{\infty}.$$

The desired identity follows from previous work of the authors and Li [62, Section 6.3].

(3) Recall that $e_{sm}(N)$ is the sum of the smallest parts of the partitions of N . We show that

$$\sum_{N=0}^{\infty} e_{sm}(N)q^N = \sum_{\lambda} sm(\lambda)q^{|\lambda|} = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q)_n}. \tag{2.2}$$

The first equality is the definition of $e_{sm}(N)$. Let $c_{lg}(\lambda)$ denote the number of occurrences of the largest part of the partition λ . Then the third member of (2.2) is equal to $\sum_{N=1}^{\infty} \left(\sum_{\lambda \in \mathcal{P}(N)} c_{lg}(\lambda) \right) q^N$. On the other hand, by looking at Young diagrams of partitions and the conjugate transformation described above, it is easy to see that $e_{sm}(N) = \sum_{\lambda \in \mathcal{P}(N)} c_{lg}(\lambda)$. Thus we deduce the second equality of (2.2). In Theorem 2.2, we put $a = b = q$ and $c = 0$ to deduce

$$\sum_{n=0}^{\infty} ((q)_n - (q)_{\infty}) = (q)_{\infty} \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q)_n},$$

which is precisely (1.7).

(4) Recall that $e_{lg}(N)$ is the sum of the largest parts of the partitions of N . By standard arguments

$$\sum_{N=0}^{\infty} e_{lg}(N)q^N = \sum_{\lambda \in \mathcal{P}} lg(\lambda)q^{|\lambda|} = \sum_{n=1}^{\infty} \frac{nq^n}{(q)_n}.$$

In previous work [62, Section 6.3] the authors and Li established

$$\sum_{n=1}^{\infty} \frac{nq^n}{(q)_n} = \frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}.$$

Equation (1.8) follows.

(5) Recall that $e_{ld}(N)$ is the sum of the largest parts of the partitions of N with distinct parts. It is clear, by definition, that

$$\sum_{N=1}^{\infty} e_{ld}(N)q^N = \sum_{\lambda \in \mathcal{D}} lg(\lambda)q^{|\lambda|} = \sum_{n=1}^{\infty} nq^n(-q)_{n-1}.$$

Equation (1.9) follows from an identity established in previous work of the authors and Li [62, Section 6.4]

$$\sum_{n=1}^{\infty} nq^n(-q)_{n-1} = (-q)_{\infty} \left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} - \frac{1}{2} \right) + \frac{1}{2}\sigma(q).$$

□

3 Circle method

This section contains the proofs of Propositions 1.8–1.10 of Sect. 1.5.

3.1 Preliminaries: a recollection on Bessel functions

We recall some standard facts about Bessel functions. For references, see [80, Chapters 6 and 7], [66, Section 10], [25,65].

The *I-Bessel function* is defined by the Schläfli's integral

$$I_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{2\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \exp\left(\frac{z^2}{4t} + t\right) dt \quad (z \in \mathbb{C} - \mathbb{R}_{\leq 0}; \nu \in \mathbb{C}). \tag{3.1}$$

In (3.1), the contour runs from $-\infty$ below the negative real axis to 0 and circles around 0 counter-clockwise and comes back to $-\infty$ above the negative real axis, and $\arg z$ is taken to be $-\pi$ below the negative real axis and π above the negative real axis (see [80, Section 6.22]). The *K-Bessel function* is defined by

$$K_\nu(z) = \frac{\pi (I_{-\nu}(z) - I_\nu(z))}{2 \sin(\pi \nu)} \quad (z \in \mathbb{C} - \mathbb{R}_{\leq 0}; \nu \in \mathbb{C} - \mathbb{Z}),$$

$$K_m(z) = \lim_{\nu \rightarrow m, \nu \notin \mathbb{Z}} K_\nu(z) \quad (z \in \mathbb{C} - \mathbb{R}_{\leq 0}; m \in \mathbb{Z}).$$

For any $\nu \in \mathbb{C}$, the pair $\{I_\nu(z), K_\nu(z)\}$ forms a basis of linearly independent solutions of the Bessel's differential equation

$$y'' + \frac{y'}{z} - \left(1 + \frac{\nu^2}{z^2}\right)y = 0.$$

Theorem 3.1 ([80, Section 7.23], [66, Section 10.40]) *Suppose that $\nu \in \mathbb{C}$ and that p is a positive integer which satisfies $p \geq \operatorname{Re} \nu - \frac{1}{2}$. Then*

$$I_\nu(z) = \frac{e^z}{(2\pi z)^{\frac{1}{2}}} \left\{ \sum_{m=0}^{p-1} \frac{(-1)^m (v, m)}{(2z)^m} + \frac{e^{-z + (v + \frac{1}{2})\pi i}}{(2\pi z)^{\frac{1}{2}}} \sum_{m=0}^{p-1} \frac{(v, m)}{(2z)^m} + O_{\nu,p}(z^{-p}) \right\} \\ \times \left(-\frac{\pi}{2} < \arg z < \frac{\pi}{2}, |z| \rightarrow \infty \right), \tag{3.2}$$

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left\{ \sum_{m=0}^{p-1} \frac{(v, m)}{(2z)^m} + O_{\nu,p}(z^{-p}) \right\} \\ \times \left(-\frac{\pi}{2} < \arg z < \frac{\pi}{2}, |z| \rightarrow \infty \right). \tag{3.3}$$

For brevity, we rewrite (3.2) and (3.3) as follows:

Theorem 3.2 *If $0 < \delta < \frac{1}{2}\pi$ and $\nu \in \mathbb{C}$, then*

$$I_{\pm\nu}(z) = \frac{e^z}{z^{\frac{1}{2}}} \left(\sum_{r=0}^{R-1} z^{-r} a_r(\nu) + O(z^{-R}) \right) \quad \left(|\arg z| < \frac{\pi}{2} - \delta, |z| \rightarrow \infty \right), \tag{3.4}$$

$$K_{\pm\nu}(z) = \frac{e^{-z}}{z^{\frac{1}{2}}} \left(\sum_{r=0}^{R-1} z^{-r} \frac{a_r(\nu)\pi}{(-1)^r} + O(z^{-R}) \right) \quad \left(|\arg z| < \frac{\pi}{2} - \delta, |z| \rightarrow \infty \right), \tag{3.5}$$

where

$$a_r(\nu) = \frac{(v, r)}{(2\pi)^{\frac{1}{2}}(-2)^r} = \frac{\left(-\frac{1}{2}\right)^r}{(2\pi)^{\frac{1}{2}}} \frac{\Gamma(\nu + 1/2 + r)}{r! \Gamma(\nu + 1/2 - r)} \quad (r \in \mathbb{Z}_{\geq 0}, \nu \in \mathbb{C}). \tag{3.6}$$

Proof Indeed, (3.5) follows from (3.3) and the fact that (ν, r) is an even function of ν , whereas (3.4) follows from (3.2), the same fact on (ν, r) , and the assumption $|\arg z| < \frac{\pi}{2} - \delta$ which forces the sum $\frac{e^{-z+(\nu+\frac{1}{2})\pi i}}{(2\pi z)^{\frac{1}{2}}} \sum_{m=0}^{p-1} \frac{(\nu, m)}{(2z)^m}$ in (3.2) to go to 0 as $|z| \rightarrow \infty$. \square

The derivative with respect to order of the I -Bessel function is $I_{\nu_0}^*(z) := \frac{\partial I_{\nu}(z)}{\partial \nu} |_{\nu=\nu_0}$. One is allowed to differentiate under the integral sign Schl\"afli's integral (3.1). Then

$$I_{\nu}^*(z) = I_{\nu}(z) \log\left(\frac{z}{2}\right) - \frac{\left(\frac{z}{2}\right)^{\nu}}{2\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \log t \exp\left(\frac{z^2}{4t} + t\right) dt \quad (z \in \mathbb{C} - \mathbb{R}_{\leq 0}; \nu \in \mathbb{C}). \tag{3.7}$$

When the order ν is an integer, we have (see [25,66, Section 10.38])

$$(-1)^n I_{\pm n}^*(z) = -K_n(z) \pm \frac{n!}{2\left(\frac{1}{2}z\right)^n} \sum_{j=0}^{n-1} (-1)^j \frac{\left(\frac{1}{2}z\right)^j I_j(z)}{j!(n-j)} \quad (z \in \mathbb{C} - \mathbb{R}_{\leq 0}; n \in \mathbb{Z}_{\geq 0}). \tag{3.8}$$

As a consequence, we derive an asymptotic expansion of $I_{\nu}^*(z)$ for integer order ν :

Theorem 3.3 *For every $0 < \delta < \frac{\pi}{2}$ one has*

$$I_0^*(z) = -K_0(z) = \frac{e^{-z}}{z^{\frac{1}{2}}} \left(\sum_{r=0}^{R-1} z^{-r} \frac{a_r(0)\pi}{(-1)^{r+1}} + O(z^{-R}) \right) \quad \left(|\arg z| < \frac{\pi}{2} - \delta, |z| \rightarrow \infty \right). \tag{3.9}$$

If $n \in \mathbb{Z}_{>0}$, for every $0 < \delta < \frac{\pi}{2}$ one has

$$I_{\pm n}^*(z) = \pm \frac{e^z}{z^{\frac{3}{2}}} \left(\sum_{r=0}^{R-1} z^{-r} a_r^*(n) + O(z^{-R}) \right) \quad \left(|\arg z| < \frac{\pi}{2} - \delta, |z| \rightarrow \infty \right), \tag{3.10}$$

where

$$a_r^*(n) = - \sum_{j=0}^{\min(n-1,r)} \frac{n!(-2)^j}{(n-1-j)!(j+1)} a_{r-j}(n-1-j) \quad (r, n \in \mathbb{Z}_{\geq 0}) \quad . \tag{3.11}$$

Proof Combining (3.8) with (3.4) and (3.5), we deduce both (3.9) and (3.10) immediately. \square

Let us define (this is a nonstandard notation)

$$D_{\nu}(z) := \frac{\left(\frac{1}{2}z\right)^{\nu}}{2\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \log t \exp\left(\frac{z^2}{4t} + t\right) dt \quad (z \in \mathbb{C} - \mathbb{R}_{\leq 0}; \nu \in \mathbb{C}), \tag{3.12}$$

so that $D_{\nu}(z) = I_{\nu}(z) \log\left(\frac{1}{2}z\right) - I_{\nu}^*(z)$.

Theorem 3.4 *If $0 < \delta < \frac{\pi}{2}$ and n is a positive integer, then*

$$D_0(z) = \frac{e^z \log\left(\frac{1}{2}z\right)}{z^{\frac{1}{2}}} \left(\sum_{r=0}^{R-1} z^{-r} a_r(0) + O(z^{-R}) \right) \times \left(|\arg z| < \frac{\pi}{2} - \delta, |z| \rightarrow \infty \right), \tag{3.13}$$

$$D_{\pm n}(z) = \frac{e^z}{z^{\frac{1}{2}}} \left(\log\left(\frac{1}{2}z\right) \sum_{r=0}^{R-1} z^{-r} a_r(n) \mp \sum_{r=0}^{R-2} z^{-r-1} a_r^*(n) + O(z^{-R} \log z) \right) \times \left(|\arg z| < \frac{\pi}{2} - \delta, |z| \rightarrow \infty \right), \tag{3.14}$$

where $a_r(0)$ is computed by (3.6) and $a_r^*(n)$ is given by (3.11).

Proof Plugging (3.4) and (3.9) into (3.7), we deduce (3.13). Plugging (3.4) and (3.10) into (3.7), we deduce (3.14). □

There are analogues for half-integer order of (3.8), Theorem 3.3, and Theorem 3.4. By the results in [25], we deduce the following asymptotic expansion of $I_\nu^*(z)$ for half-integer order ν :

Theorem 3.5 (1) *If R is a positive integer, then*

$$I_{-\frac{1}{2}}^*(x) = \frac{e^x}{x^{\frac{3}{2}}} \left(\sum_{r=0}^{R-1} x^{-r} \frac{r!}{2^{r+\frac{3}{2}} \pi^{\frac{1}{2}}} + O(x^{-R}) \right) \quad (x \in \mathbb{R}_{>0}, x \rightarrow \infty). \tag{3.15}$$

(2) *If n is a positive integer, then*

$$I_{-n-\frac{1}{2}}^*(x) = O\left(\frac{e^x}{x^{\frac{1}{2}}}\right) \quad (x \in \mathbb{R}_{>0}, x \rightarrow \infty). \tag{3.16}$$

3.2 Polynomial-type case

We are in a position to prove Proposition 1.8. In this subsection we make available all the hypotheses stated therein. Set

$$V_s(N) = \frac{1}{2\pi i} \int_{C_1} \frac{t^{s-B}}{q^{N+1}} \cdot t^\beta \exp\left(\frac{c^2}{t}\right) dq \tag{3.17}$$

where q describes the arc C_1 counter-clockwise.

The proof of Proposition 1.8 rests on the following two estimates:

Lemma 3.6 *Let k be a positive integer. One has*

$$V(N) = \sum_{s=0}^{k-1} \alpha_s V_s(N) + O\left(e^{2c\sqrt{N}} N^{\frac{1}{2}(B-\beta-k-1)}\right) \quad (N \rightarrow \infty). \tag{3.18}$$

Lemma 3.7 *Let $s \geq 0$ and $R \geq 1$ be integers. One has*

$$V_s(N) = e^{2c\sqrt{N}} N^{\frac{1}{4}(2B-2s-2\beta-3)} \left(\sum_{r=0}^{R-1} w_{s,r} N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right) \right) \quad (N \rightarrow \infty), \tag{3.19}$$

where $w_{s,r}$ is given by (1.16).

Remark An acute reader will observe that the r th-main term of $V_s(N)$, i.e., the term in which $w_{s,r}$ occurs, dominates the error term of $V(N)$ when $s + r < k$.

Lemmas 3.6 and 3.7 are established by Wright [81, Lemmas 2 and 3]. We sketch Wright’s proofs of these estimates below not only for the sake of completeness but also to recycle his approach in proving Propositions 1.9 and 1.10.

Proof of Lemma 3.6 As in [81], we have $V - \sum_{s=0}^{k-1} \alpha_s V_s = E_1 + E_2 + E_3$, where

$$\begin{aligned}
 E_1 &= \frac{1}{2\pi i} \int_{C_2} \frac{L\xi}{q^{N+1}} dq, \\
 E_2 &= \frac{1}{2\pi i} \int_{C_1} \frac{L\left(\xi - t^\beta \exp\left(\frac{c^2}{t}\right)\right)}{q^{N+1}} dq, \\
 E_3 &= \frac{1}{2\pi i} \int_{C_1} \frac{\left(L - \sum_{s=0}^{k-1} \alpha_s t^{s-B}\right) \cdot t^\beta \exp\left(\frac{c^2}{t}\right)}{q^{N+1}} dq;
 \end{aligned}$$

here the contours are described counter-clockwise.

Recall that $\eta = \frac{c}{\sqrt{N}}$ and that $q = e^{-t}$ is on the circle ($C : |q| = e^{-\eta}$). Therefore $\text{Re } t = \eta$. If q is on the arc C_1 , we further have $\text{Im}(t) \ll_\delta \eta$.

To prove Lemma 3.6, it suffices to show the following upper bounds:

$$\begin{aligned}
 E_1 &\ll N^{C-\frac{\beta}{2}} \exp((2c - \delta')\sqrt{N}), \\
 E_2 &\ll N^{\frac{1}{2}(B-\beta-1)} \exp(c\sqrt{N}), \\
 E_3 &\ll N^{\frac{1}{2}(B-k-1-\beta)} \exp(2c\sqrt{N}).
 \end{aligned}$$

First, for E_3 observe that $L(e^{-t}) - \sum_{s=0}^{k-1} \alpha_s t^{s-B} \ll t^{k-B} \asymp \eta^{k-B}$ by HYPO 1 and that $\left|\exp\left(\frac{c^2}{t} + Nt\right)\right| \leq \exp\left(2c\sqrt{N}\right)$. Since the length of C_1 is $\asymp \eta$, we have

$$E_3 \ll \eta^{k+1+\beta-B} \exp(2c\sqrt{N}) \asymp N^{\frac{1}{2}(B-k-1-\beta)} \exp(2c\sqrt{N}).$$

Second, for E_2 we note that $L(e^{-t}) \ll t^{-B} \asymp \eta^{-B}$ by HYPO 1 and that $\xi(e^{-t}) - t^\beta \exp\left(\frac{c^2}{t}\right) \ll \eta^\beta \exp\left(\frac{c^2-\gamma}{t}\right) \ll \eta^\beta$ by HYPO 2, whence

$$E_2 \ll \eta^{\beta+1-B} \exp(Nt) \ll \eta^{\beta+1-B} \exp\left(c\sqrt{N}\right) \asymp N^{\frac{1}{2}(B-\beta-1)} \exp(c\sqrt{N}).$$

Finally, we use the assumptions on the “minor arc” C_2 to establish the bound for E_1 . By HYPO 3, for $q \in C_2$ we have $L(q) \ll N^C$. By HYPO 4, for $q \in C_2$ we have $|\xi(q)| \ll \xi(|q|) \exp(-\delta'\sqrt{N})$. Since $|q| \in C_1$, we have

$$\frac{\xi(|q|)}{q^N} \ll \eta^\beta \exp\left(\frac{c^2}{\eta} + N\eta\right) \asymp N^{-\frac{\beta}{2}} \exp\left(2c\sqrt{N}\right).$$

Thus $E_1 \ll N^{C-\frac{\beta}{2}} \exp\left((2c - \delta')\sqrt{N}\right)$.

On combining the three bounds for E_1, E_2, E_3 we conclude that

$$V(N) - \sum_{s=0}^{k-1} \alpha_s V_s(N) \ll N^{\frac{1}{2}(B-k-1-\beta)} \exp(2c\sqrt{N}) \quad (N \rightarrow \infty).$$

□

Proof of Lemma 3.7 As in [81], we let \mathcal{D}_2 denote the line on the t -plane such that $q = e^{-t} \in \mathcal{C}_1$: this means $\operatorname{Re} t = \eta$ and $|\operatorname{Im} t| \leq A\eta$ where $A = A(\delta)$ is a positive constant. Let \mathcal{D}_1 (resp. \mathcal{D}_3) denote the line on which $\operatorname{Re} t \leq \eta$ and $\operatorname{Im} t = -A\eta$ (resp. $\operatorname{Im} t = A\eta$). Write $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3$ for the contour with counter-clockwise orientation. By (3.17), we have

$$V_s(N) = \frac{1}{2\pi i} \int_{\mathcal{D}_2} t^{s+\beta-B} \exp\left(Nt + \frac{c^2}{t}\right) dt.$$

Set

$$W_s(N) = \frac{1}{2\pi i} \int_{\mathcal{D}} t^{s+\beta-B} \exp\left(Nt + \frac{c^2}{t}\right) dt.$$

We claim that W_s is a good approximation for V_s , namely:

$$V_s(N) - W_s(N) \ll \exp\left(\frac{3}{2}c\sqrt{N}\right). \tag{3.20}$$

Indeed, if t lies on \mathcal{D}_3 , then $\operatorname{Re}\left(\frac{c^2}{t}\right) = c^2 \frac{\operatorname{Re} t}{|t|^2} \leq \frac{c\sqrt{N}}{2}$ and, on putting $t = (\eta - i \cdot A\eta) - u$,

$$\left| t^{s+\beta-B} \exp\left(Nt + \frac{c^2}{t}\right) \right| \ll \left(1 + u^{s+\beta-B}\right) \exp\left(\frac{3}{2}c\sqrt{N} - Nu\right).$$

The same bound holds when t lies on \mathcal{D}_1 . Thus

$$\begin{aligned} V_s(N) - W_s(N) &\ll \exp\left(\frac{3c}{2}\sqrt{N}\right) \int_0^\infty (1 + u^{s+\beta-B}) \exp(-Nu) du \\ &\ll \exp\left(\frac{3}{2}c\sqrt{N}\right), \end{aligned}$$

so our claim (3.20) is proven.

On the other hand, by (3.1) we have

$$W_s(N) = \left(\frac{c^2}{N}\right)^{\frac{1}{2}(s+\beta-B+1)} I_{B-\beta-s-1}(2c\sqrt{N}). \tag{3.21}$$

By appealing to asymptotic expansion (3.4) of the I -Bessel function, we see that

$$W_s(N) = e^{2c\sqrt{N}} N^{\frac{1}{4}(2B-2s-2\beta-3)} \left(\sum_{r=0}^{R-1} N^{-\frac{r}{2}} w_{s,r} + O\left(N^{-\frac{R}{2}}\right) \right) \quad (N \rightarrow \infty), \tag{3.22}$$

where

$$\begin{aligned} w_{s,r} &= \frac{c^{s+\beta-B+1}}{(2c)^{r+\frac{1}{2}}} a_r(s + \beta - B + 1) \\ &= \frac{\left(-\frac{1}{4c}\right)^r c^{s+\beta-B+\frac{1}{2}}}{2\pi^{\frac{1}{2}}} (s + \beta - B + 1, r). \end{aligned}$$

Finally, on combining (3.20) and (3.22), we conclude the lemma. □

Proof of Proposition 1.8 As in [81], on taking $k = R + 1$ in asymptotic (3.18) of Lemma 3.6 and taking $s = R$ in asymptotic (3.19) of Lemma 3.7, we see that

$$V(N) - \sum_{s=0}^{R-1} \alpha_s V_s(N) \ll e^{2c\sqrt{N}} N^{\frac{1}{4}(2B-2R-2\beta-3)} \quad (N \rightarrow \infty). \tag{3.23}$$

By Lemma 3.7, each V_s admits the asymptotic expansion

$$\begin{aligned} V_s(N) &= e^{2c\sqrt{N}} N^{\frac{1}{4}(2B-2s-2\beta-3)} \left(\sum_{r=0}^{R-s-1} w_{s,r} N^{-\frac{r}{2}} + O\left(N^{-\frac{R-s}{2}}\right) \right) \\ &= e^{2c\sqrt{N}} N^{\frac{1}{4}(2B-2\beta-3)} \left(\sum_{r=s}^{R-1} w_{s,r-s} N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right) \right) \quad (N \rightarrow \infty). \end{aligned} \tag{3.24}$$

On plugging (3.24) into (3.23), we conclude Proposition 1.8. □

3.3 Logarithmic-type case

We are in a position to prove Propositions 1.9 and 1.10. In this subsection we make available all the hypotheses stated therein, except for the condition $B - \beta = 1$ or $B - \beta = \frac{1}{2}$. Set

$$V_s^*(N) = \frac{1}{2\pi i} \int_{C_1} \frac{t^{s-B} \log t}{q^{N+1}} t^\beta \exp\left(\frac{c^2}{t}\right) dq. \tag{3.25}$$

The proof of Proposition 1.9 rests on the following two estimates:

Lemma 3.8 *Let k be a positive integer. The condition on $B - \beta$ can be omitted. One has*

$$V(N) = \sum_{s=0}^{k-1} \alpha_s V_s^*(N) + O\left(e^{2c\sqrt{N}} N^{\frac{1}{2}(B-\beta-k-1)} \log N\right) \quad (N \rightarrow \infty). \tag{3.26}$$

Lemma 3.9 *Let $s \geq 0$ and $R \geq 1$ be integers. Suppose $B - \beta = 1$. One has*

$$V_s^*(N) = e^{2c\sqrt{N}} N^{-\frac{1}{4}(2s+1)} \left(\sum_{r=0}^{R-1} N^{-\frac{r}{2}} (l'_{s,r} + l_{s,r} \log N) + O(N^{-\frac{R}{2}} \log N) \right) \quad (N \rightarrow \infty), \tag{3.27}$$

where $l_{s,r}, l'_{s,r}$ are given in Proposition 1.9.

We will establish Lemmas 3.8 and 3.9 similarly to Lemmas 3.6 and 3.7.

Proof of Lemma 3.8 Similar to Lemma 3.6, we have $V - \sum_{s=0}^{k-1} V_s^* = E_1 + E_2 + E_3$, where

$$\begin{aligned} E_1 &= \frac{1}{2\pi i} \int_{C_2} \frac{L\xi}{q^{N+1}} dq, \\ E_2 &= \frac{1}{2\pi i} \int_{C_1} \frac{L\left(\xi - t^\beta \exp\left(\frac{c^2}{t}\right)\right)}{q^{N+1}} dq, \\ E_3 &= \frac{1}{2\pi i} \int_{C_1} \frac{\left(L - \frac{\log t}{t^B} \left(\sum_{s=0}^{k-1} \alpha_s t^s\right)\right) \cdot t^\beta \exp\left(\frac{c^2}{t}\right)}{q^{N+1}} dq; \end{aligned}$$

here the contours are described counter-clockwise.

Recall that $\eta = \frac{c}{\sqrt{N}}$ and that $q = e^{-t}$ is on the circle ($\mathcal{C} : |q| = e^{-\eta}$). Therefore $\operatorname{Re} t = \eta$. If q is on the arc \mathcal{C}_1 , we further have $\operatorname{Im}(t) \ll_{\delta} \eta$.

To prove Lemma 3.8, it suffices to show the following upper bounds:

$$\begin{aligned} E_1 &\ll \exp((2c - \delta')\sqrt{N})N^C, \\ E_2 &\ll \exp(c\sqrt{N}) \log N, \\ E_3 &\ll \exp(2c\sqrt{N})N^{\frac{1}{2}(B-k-1-\beta)} \log N. \end{aligned}$$

First, for E_3 observe that $L(e^{-t}) - \frac{\log t}{t^B}(\sum_{s=0}^{k-1} \alpha_s t^s) \ll t^{k-B} |\log t| \asymp \eta^{k-B} \log N$ by HYPO 1 and that $\left| \exp\left(\frac{c^2}{t} + Nt\right) \right| \leq \exp\left(2c\sqrt{N}\right)$. Since the length of \mathcal{C}_1 is $\asymp \eta$, we see that

$$E_3 \ll \eta^{k+1+\beta-B} \exp(2c\sqrt{N}) \log N \asymp N^{\frac{1}{2}(B-k-1-\beta)} \exp(2c\sqrt{N}) \log N.$$

Second, for E_2 we note that $L(e^{-t}) \ll \frac{\log \eta}{\eta^B} \asymp \eta^{-B} \log N$ by HYPO 1 and that $\xi - t^{\beta} \exp\left(\frac{c^2}{t}\right) \ll t^{\beta} \exp\left(\frac{c^2 - \gamma}{t}\right) \ll \eta^{\beta}$ by HYPO 2. Therefore

$$E_2 \ll \eta^{\beta-B} \exp\left(Nt + \frac{c^2 - \gamma}{t}\right) \log N \ll \eta^{\beta-B} \exp\left(c\sqrt{N}\right) \log N.$$

Finally, on using the assumptions on the growth of L and ξ on the “minor arc” \mathcal{C}_2 , we establish the bound for E_1 . By HYPO 3, for $q \in \mathcal{C}_2$ we have $L(q) \ll N^C$. By HYPO 4, for $q \in \mathcal{C}_2$ we have $\xi(q) \ll \xi(|q|) \exp(-\delta'\sqrt{N})$. Since $|q| \in \mathcal{C}_1$ we have

$$\left| \frac{\xi(|q|)}{q^N} \right| \ll \exp\left(\frac{c^2}{\eta} + N\eta\right) = \exp\left(2c\sqrt{N}\right).$$

Thus $E_1 \ll N^C \exp\left((2c - \delta')\sqrt{N}\right)$.

On combining the three bounds for E_1, E_2, E_3 we arrive at

$$V - \sum_{s=0}^{k-1} \alpha_s V_s^* \ll N^{\frac{1}{2}(B-k-1-\beta)} \exp(2c\sqrt{N}) \log N.$$

□

Proof of Lemma 3.9 As in Lemma 3.7, let \mathcal{D}_2 denote the line on the t -plane on which $\operatorname{Re} t = \eta$ and $|\operatorname{Im} t| \leq A\eta$. Let \mathcal{D}_1 (resp. \mathcal{D}_3) denote the line on which $\operatorname{Re} t \leq \eta$ and $\operatorname{Im} t = -A\eta$ (resp. $\operatorname{Im} t = A\eta$). Write $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3$ for the contour with counter-clockwise orientation. By (3.25) we have

$$V_s^*(N) = \frac{1}{2\pi i} \int_{\mathcal{D}_2} t^{s-1} \log t \cdot \exp\left(\frac{c^2}{t} + Nt\right) dt.$$

Set

$$W_s^*(N) = \frac{1}{2\pi i} \int_{\mathcal{D}} t^{s-1} \log t \cdot \exp\left(\frac{c^2}{t} + Nt\right) dt.$$

We claim that W_s^* is a good approximation for V_s^* , namely:

$$V_s^*(N) - W_s^*(N) \ll \exp\left(\frac{3}{2}c\sqrt{N}\right) \cdot \log N. \tag{3.28}$$

Indeed, if t lies on \mathcal{D}_3 , then $\operatorname{Re}\left(\frac{c^2}{t}\right) \leq \frac{c^2}{2\eta} = \frac{c\sqrt{N}}{2}$ and, on putting $t = (\eta - i \cdot A\eta) - u$,

$$\left| t^{s-1} \log t \cdot \exp\left(Nt + \frac{c^2}{t}\right) \right| \ll (1 + u^{s-1}) \log(Nu) \exp\left(\frac{3}{2}c\sqrt{N} - Nu\right).$$

The same bound holds when t lies on \mathcal{D}_1 . Thus

$$\begin{aligned} V_s^*(N) - W_s^*(N) &\ll \exp\left(\frac{3}{2}c\sqrt{N}\right) \int_0^\infty (1 + u^{s-1}) \log(Nu) \exp(-Nu) du \\ &\ll \exp\left(\frac{3}{2}c\sqrt{N}\right) \cdot \log N, \end{aligned}$$

so our claim is proven.

On the other hand, by (3.1) and (3.12) we have

$$W_s^*(N) = \left(\frac{c^2}{N}\right)^{\frac{1}{2}s} \left(D_{-s}(2c\sqrt{N}) - I_{-s}(2c\sqrt{N}) \log N\right). \tag{3.29}$$

It now follows from (3.29), (3.4), and Theorem 3.4 that

$$W_s^*(N) = e^{2c\sqrt{N}} N^{-\frac{1}{4}(2s+1)} \left(\sum_{r=0}^{R-1} N^{-\frac{r}{2}} (l'_{s,r} + l_{s,r} \log N) + O(N^{-\frac{R}{2}} \log N)\right) (N \rightarrow \infty). \tag{3.30}$$

Here we combine (3.4), (3.13), and (3.29) to get the formulas for the coefficients $l'_{s,r}$ and $l_{s,r}$:

$$\begin{aligned} l_{s,r} &= \frac{\left(-\frac{1}{4c}\right)^r c^{s-\frac{1}{2}}}{-4\pi^{\frac{1}{2}}} (s, r) && (s, r \in \mathbb{Z}_{\geq 0}), \\ l'_{s,r} &= \frac{\left(-\frac{1}{4c}\right)^r c^{s-\frac{1}{2}} \log c}{2\pi^{\frac{1}{2}}} (s, r) + \frac{c^{s-r-\frac{1}{2}}}{2^{r+\frac{1}{2}}} a_{r-1}^*(s) && (s, r \in \mathbb{Z}_{\geq 0}). \end{aligned}$$

Finally, on combining (3.28) and (3.30) we conclude the lemma. □

Proof of Proposition 1.9 As in the proof of Proposition 1.8, on taking $k = R + 1$ in Lemma 3.8 and taking $s = R$ in Lemma 3.9 we deduce that

$$V(N) - \sum_{s=0}^{R-1} \alpha_s V_s^*(N) \ll e^{2c\sqrt{N}} N^{-\frac{1}{4}(2R+1)} \log N \quad (N \rightarrow \infty). \tag{3.31}$$

Then we use Lemma 3.9 to plug in the asymptotic expansion of each V_s^* , namely

$$\begin{aligned} V_s^*(N) &= e^{2c\sqrt{N}} N^{-\frac{1}{4}(2s+1)} \left(\sum_{r=0}^{R-s-1} N^{-\frac{r}{2}} (l'_{s,r} + l_{s,r} \log N) + O(N^{-\frac{1}{2}(R-s)} \log N)\right) \\ &= e^{2c\sqrt{N}} N^{-\frac{1}{4}} \left(\sum_{r=s}^{R-1} N^{-\frac{r}{2}} (l'_{s,r-s} + l_{s,r-s} \log N) + O(N^{-\frac{R}{2}} \log N)\right). \end{aligned} \tag{3.32}$$

On plugging (3.32) into (3.31), we conclude Proposition 1.9. □

Proof of Proposition 1.10 By appealing to Theorems 3.2 and 3.5, one deduces the following estimate:

Lemma 3.10 *Let $s \geq 1$ and $R \geq 1$ be integers. If $B - \beta = \frac{1}{2}$, then*

$$V_0^*(N) = e^{2c\sqrt{N}} N^{-\frac{1}{2}} \cdot \frac{-1}{4\pi^{\frac{1}{2}}} \left(\log N - 2 \log c + O(N^{-\frac{1}{2}} \log N) \right) \quad (N \rightarrow \infty),$$

$$V_s^*(N) = O\left(e^{2c\sqrt{N}} N^{-\frac{s}{2} - \frac{1}{2}} \log N \right) \quad (N \rightarrow \infty).$$

The proof of Proposition 1.10 then rests on Lemmas 3.8 and 3.10. By the same token as the proof of Proposition 1.9, one shows Proposition 1.10 in a straightforward manner. \square

4 Asymptotic expansion

In this section we compute various asymptotic expansions of holomorphic modular forms and quantum modular forms. These functions are the building blocks of the generating functions of Theorems 1.1–1.7.

4.1 The partition function and the Dedekind eta function

A pivoting character in the theory of combinatorial analysis of partition functions is the Dedekind’s eta function

$$\eta(z) = q^{\frac{1}{24}} (q)_\infty \quad (z \in \mathbb{H}; q = e^{2\pi iz}). \tag{4.1}$$

It is a half-integral weight modular form. Its relation to the partition function stems from the identity

$$\xi_\infty(q) := \frac{1}{(q)_\infty} = \sum_{N=1}^\infty p(N)q^N.$$

The modularity of the Dedekind eta function is a key ingredient in the circle method which provides asymptotic expansion for the partition function $p(N)$. According to Hardy–Ramanujan and Wright, among others (see [49, Section 1.4, p. 280] and [81, pp. 112–113]):

Theorem 4.1 *Let $0 < \delta < \frac{\pi}{2}$ be a constant.*

(1) *As $|t| \rightarrow 0$ in the bounded cone $|\arg t| < \frac{\pi}{2} - \delta$*

$$\xi_\infty(e^{-t}) = \left(\frac{t}{2\pi} \right)^{\frac{1}{2}} e^{\frac{\pi^2}{6t} - \frac{t}{24}} \left(1 + O_\delta \left(e^{-\frac{4\pi^2}{t}} \right) \right),$$

$$\frac{1}{\eta(e^{-t})} = \left(\frac{t}{2\pi} \right)^{\frac{1}{2}} e^{\frac{\pi^2}{6t}} \left(1 + O_\delta \left(e^{-\frac{4\pi^2}{t}} \right) \right).$$

(2) *There exists a positive constant $\delta' = \delta'(\delta)$ such that, as $|t| \rightarrow 0$ in the bounded cone $\frac{\pi}{2} - \delta \leq |\arg t| < \frac{\pi}{2}$,*

$$\xi_\infty(e^{-t}) \ll_\delta \xi_\infty(|e^{-t}|) \exp\left(\frac{-\delta'}{t}\right),$$

$$\frac{1}{\eta(e^{-t})} \ll_\delta \frac{1}{\eta(|e^{-t}|)} \exp\left(\frac{-\delta'}{t}\right).$$

4.2 The divisor function and a period function

In their recent work [13], Bettin and Conrey studied the divisor generating function

$$S_0(z) = \sum_{n=1}^{\infty} d(n)e^{2\pi inz} \quad (z \in \mathbb{H}),$$

where $d(n)$ denotes the number of positive divisors of n and \mathbb{H} denotes the Poincaré upper half-plane. Bettin and Conrey discovered that $S_0(z)$ is associated with the period function $\psi_0(z)$ given by

$$\psi_0(z) = \left(1 - \frac{1}{z}\right) - 4 \left(S_0(z) - \frac{1}{z}S_0\left(-\frac{1}{z}\right)\right),$$

and showed the following

Theorem 4.2 (Bettin–Conrey) [13, Theorem 1] *The period function $\psi_0(z)$, originally defined for $z \in \mathbb{H}$, extends to an analytic function on $\mathbb{C}' := \mathbb{C} - \mathbb{R}_{\leq 0}$ via the representation*

$$\psi_0(z) = \frac{-2(\log(2\pi z) - \gamma)}{\pi iz} - 2ig_0(z) \tag{4.2}$$

where $\gamma = 0.5772156\dots$ is Euler’s constant and

$$g_0(z) = 2 \sum_{n=1}^M \frac{(-1)^{n+1} B_{2n}}{(2n)!} \zeta(1 - 2n)(2\pi z)^{2n-1} + \frac{1}{\pi i} \int_{(-\frac{1}{2}-2M)} \zeta(s)^2 \frac{\Gamma(s)}{\sin \frac{\pi s}{2}} (2\pi z)^{-s} ds \tag{4.3}$$

for any nonnegative integer M .

Our main goal in this paragraph is to show that the function $L(e^{-t}) := \psi_0\left(\frac{it}{2\pi}\right)$ is a linear combination of a function of polynomial type and a function of logarithmic type (see HYPO 1 in Sect. 1.5). We first start with the term $g_0(z) = g_0\left(\frac{it}{2\pi}\right)$. It is clear from (4.3) that

$$g_0\left(\frac{it}{2\pi}\right) = -i \sum_{n=1}^M \frac{B_{2n}^2}{(2n)!n} t^{2n-1} + \frac{1}{\pi i} \int_{(-\frac{1}{2}-2M)} \zeta^2(s) \frac{\Gamma(s)}{\sin \frac{\pi s}{2}} (it)^{-s} ds \quad (\text{Re } t > 0).$$

Set

$$A_M(t) = \frac{1}{\pi i} \int_{(-\frac{1}{2}-2M)} \zeta^2(s) \frac{\Gamma(s)}{\sin \frac{\pi s}{2}} (it)^{-s} ds \quad (\text{Re } t > 0).$$

Lemma 4.3 *Let M be a positive integer. As $|t| \rightarrow 0$ in the right half-plane $\text{Re } t > 0$, one has $A_M(t) \ll_M t^{2M+\frac{1}{2}}$.*

Proof Using the functional equation of the Riemann zeta function $\zeta(s)$ together with the identities $\Gamma(s)\Gamma(s + \frac{1}{2}) = \pi^{\frac{1}{2}} 2^{1-2s} \Gamma(2s)$ and $\Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin(\pi s)}$, we see that

$$\zeta^2(s) \frac{\Gamma(s)}{\sin \frac{\pi s}{2}} (it)^{-s} = \frac{(2\pi)^{2s-1}}{\cos \frac{\pi s}{2}} \Gamma(1 - s) \zeta^2(1 - s) (it)^{-s}.$$

Therefore

$$A_M(t) := \frac{1}{\pi i} \int_{(\frac{1}{2}+2M)} \frac{(2\pi)^{-2s-1}}{\cos \frac{\pi s}{2}} \Gamma(1 + s) \zeta^2(1 + s) (it)^{-s} ds \quad (\text{Re } t > 0).$$

Write $s = \frac{1}{2} + 2M + it_0$. We break the integral $\int_{(\frac{1}{2}+2M)} ds = i \int_{-\infty}^{\infty} dt_0$ into three parts: (a) $|t_0| \geq 220M$, (b) $1 \leq |t_0| < 220M$, and (c) $|t_0| < 1$ and hence write $A = A_a + A_b + A_c$ accordingly.

On part (a), i.e., $|t_0| \geq 220M$ and $\sigma = \frac{1}{2} + 2M$, we have the following estimates

$$\begin{aligned} (2\pi)^{-2s-1} &\sim (2\pi)^{-4M-2} \\ \frac{1}{\cos \frac{\pi s}{2}} &\asymp e^{-\frac{1}{2}\pi t_0} \\ \Gamma(1+s) &\asymp t_0^{1+2M} e^{-\frac{1}{2}\pi t_0} \quad (\text{by Stirling's formula}) \\ \zeta^2(1+s) &\ll_M 1 \\ (it)^{-s} &= e^{\frac{1}{2}\pi i s} t^s \sim e^{-\frac{1}{2}\pi t_0} t^{\frac{1}{2}+2M}; \end{aligned}$$

here all estimates except for the second line are apparent, whereas for the second line we note

$$2 \cos \frac{\pi s}{2} = e^{\frac{1}{2}\pi i s} + e^{-\frac{1}{2}\pi i s} = e^{\frac{1}{2}\pi i \sigma} e^{-\frac{1}{2}\pi t_0} + e^{-\frac{1}{2}\pi i \sigma} e^{\frac{1}{2}\pi t_0} \asymp e^{\frac{1}{2}\pi t_0}.$$

Hence $A_a \ll_M t^{\frac{1}{2}+2M} \int_M^{\infty} t_0^{1+2M} e^{-\frac{3}{2}\pi t_0} dt_0 \ll_M t^{\frac{1}{2}+2M}$. Similar arguments show that $A_b \ll_M t^{\frac{1}{2}+2M}$ and that $A_c \ll_M t^{\frac{1}{2}+2M}$. This concludes the lemma. \square

Corollary 4.4 *Let M be a positive integer. One has*

$$\begin{aligned} g_0\left(\frac{it}{2\pi}\right) &= -i \sum_{n=1}^M \frac{B_{2n}^2}{(2n)! n} t^{2n-1} + O(t^{2M+\frac{1}{2}}) \quad (\text{Re } t > 0, |t| \rightarrow 0). \\ \psi_0\left(\frac{it}{2\pi}\right) &= \frac{4 \log t}{t} + \frac{2\pi i - 4\gamma}{t} - 2 \sum_{n=1}^M \frac{B_{2n}^2}{(2n)! n} t^{2n-1} + O(t^{2M+\frac{1}{2}}) \quad (\text{Re } t > 0, |t| \rightarrow 0). \end{aligned}$$

From the relation between the divisor generating function $S_0(z)$ and the associated period function $\psi_0(z)$, namely

$$S_0(z) = \frac{1}{4} \left(1 - \frac{1}{z}\right) - \frac{1}{4} \psi_0(z) + \frac{4}{z} S_0\left(-\frac{1}{z}\right),$$

we are now ready to see that the function $S_0\left(\frac{it}{2\pi}\right)$ satisfies HYPO 1 and HYPO 3.

Corollary 4.5 (1) *Let M be a positive integer. As $|t| \rightarrow 0$ in the bounded cone $|\arg t| < \frac{\pi}{2} - \delta$, one has*

$$S_0\left(\frac{it}{2\pi}\right) = -\frac{\log t}{t} + \frac{\gamma}{t} + \frac{1}{4} + \sum_{n=1}^M \frac{B_{2n}^2}{(2n)! (2n)} t^{2n-1} + O(t^{2M+\frac{1}{2}}). \tag{4.4}$$

(2) *As $|t| \rightarrow 0$ in the bounded cone $\frac{\pi}{2} - \delta \leq |\arg t| < \frac{\pi}{2}$, one has*

$$S_0\left(\frac{it}{2\pi}\right) \ll_{\delta} t^{-\frac{3}{2}}. \tag{4.5}$$

The corollary shows that the function $S_0\left(\frac{it}{2\pi}\right)$ is a sum of a series of polynomial type and a series of logarithmic type in the sense of HYPO 1, and that $S_0\left(\frac{it}{2\pi}\right)$ satisfies HYPO 3.

Proof (1) This is an immediate consequence of Corollary 4.4 and the observation that in the given bounded cone, the function $S_0\left(-\frac{1}{z}\right) = S_0\left(\frac{2\pi i}{t}\right)$ decays exponentially with respect to $\frac{1}{|t|}$.

(2) We have

$$S_0\left(-\frac{1}{z}\right) = S_0\left(\frac{2\pi i}{t}\right) = \sum_{n=1}^{\infty} d(n)e^{-\frac{4\pi^2 n}{t}}$$

$$\ll \sum_{n=1}^{\infty} n^{\epsilon} e^{-\frac{4\pi^2 n \operatorname{Re} t}{|t|^2}} \leq \sum_{n=1}^{\infty} e^{\epsilon \log n - \frac{2\pi^2 n}{N^{1/2}}}.$$

We then break this latter sum as $\sum_{n=1}^{\infty} = \sum_{n < N^{2/3}} + \sum_{n \geq N^{2/3}}$. For $n < N^{2/3}$, we have $\sum_{n < N^{2/3}} e^{\epsilon \log n - \frac{2\pi^2 n}{N^{1/2}}} \leq \sum_{n < N^{2/3}} e^{\frac{2}{3}\epsilon \log n} \leq N^{\frac{2}{3} + \frac{2\epsilon}{3}} \ll N^{\frac{3}{4}}$. For $n \geq N^{2/3}$, we have $\epsilon \log n < \frac{n}{N^{1/2}}$ and hence

$$\sum_{n \geq N^{2/3}} e^{\epsilon \log n - \frac{2\pi^2 n}{N^{1/2}}} \leq \sum_{n \geq N^{2/3}} e^{\frac{n}{N^{1/2}}(1-2\pi^2)} \leq \sum_{n \geq N^{2/3}} e^{-\frac{n}{N^{1/2}}} \ll N^{1/2}.$$

This concludes the second statement of the corollary. □

4.3 The Ramanujan–Andrews–Dyson–Hickerson–Cohen pair (σ, σ^*)

The pair (σ, σ^*) given by

$$\sigma(q) := 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (1-q)(1-q^2) \cdots (1-q^n) \quad (|q| < 1)$$

$$\sigma^*(q) := -2 \sum_{n=0}^{\infty} q^{n+1} (1-q^2)(1-q^4) \cdots (1-q^{2n}) \quad (|q| < 1)$$

is one of the very first examples in the theory of quantum modular form [83]. Set

$$R(z) := q\sigma(q^{24}) =: \sum_{n>0} T(n)q^n \quad (z \in \mathbb{H}; q = e^{2\pi iz}, |q| < 1) \tag{4.6}$$

$$R^*(z) := q^{-1}\sigma^*(q^{24}) =: \sum_{n>0} T(-n)q^n \quad (z \in \mathbb{H}; q = e^{2\pi iz}, |q| < 1). \tag{4.7}$$

Put $R_+ := R + R^*$ and $R_- := R - R^*$.

In this paragraph, let $\epsilon \in \{+, -\}$ and let $A > 1$ be any (fixed) constant. Our goal is to work out an integral representation for $R_{\epsilon}(z)$ which is analogous to (4.2) and which implies useful asymptotic expansions. Our starting observation is that $R_{\epsilon}(z)$ is an inverse Mellin transform of a certain Hecke L -function. We will look at the period function of type $\psi_{\epsilon}(z) = R_{\epsilon}(z) - \frac{1}{cz}R_{\epsilon}\left(-\frac{1}{cz}\right)$ where $c > 0$ is a normalizing constant.

Define

$$L_+(s) = \sum_{n \geq 1} \frac{T(n) + T(-n)}{n^s}; \quad \Lambda_+(s) = (1152)^{\frac{s}{2}} \pi^{-s} \Gamma\left(\frac{s}{2}\right)^2 L_+(s) \tag{4.8}$$

$$L_-(s) = \sum_{n \geq 1} \frac{T(n) - T(-n)}{n^s}; \quad \Lambda_-(s) = (1152)^{\frac{s+1}{2}} \pi^{-(s+1)} \Gamma\left(\frac{s+1}{2}\right)^2 L_-(s). \tag{4.9}$$

We collect the relevant results from [7, 16, 29] in the following theorem.

Theorem 4.6 [7, 16, 29]

- (1) $T(\epsilon n) = O(\log n)$ for $\epsilon \in \{+, -\}$ and $n \in \mathbb{Z}_{>0}$;
- (2) $L_\epsilon(s)$ converges absolutely for $\text{Re } s > 1$;
- (3) $\Lambda_\epsilon(s)$ extends to an entire function of order 1 and satisfies $\Lambda_\epsilon(s) = \Lambda_\epsilon(1 - s)$;
- (4) Outside the critical strip $0 < \text{Re } s < 1$, $L_+(s)$ has trivial zeros (of order 2) at $s \in 2\mathbb{Z}_{\leq 0} = \{0, -2, -4, -6, \dots\}$. Outside the critical strip $0 < \text{Re } s < 1$, $L_-(s)$ has trivial zeros (of order 2) at $s \in -1 + 2\mathbb{Z}_{\leq 0} = \{-1, -3, -5, -7, \dots\}$.

The following is an analogue of Theorem 4.2.

Theorem 4.7 For $z \in \mathbb{H}$ one has

$$\begin{aligned}
 R_+(z) &= \frac{1}{1152^{\frac{1}{2}} z} R_+ \left(-\frac{1}{1152z} \right) \\
 &= \sum_{n=0}^M r_{+,n} + \frac{1}{2\pi i} \int_{(-\frac{1}{2}-2M)} \frac{i}{\sin \frac{\pi s}{2}} (2\pi z)^{-s} \Gamma(s) L_+(s) ds
 \end{aligned} \tag{4.10}$$

$$\begin{aligned}
 R_-(z) &= \frac{1}{1152^{\frac{1}{2}} z} R_- \left(-\frac{1}{1152z} \right) \\
 &= \sum_{n=0}^M r_{-,n} + \frac{1}{2\pi i} \int_{(-\frac{1}{2}-2M)} \frac{1}{\cos \frac{\pi s}{2}} (2\pi z)^{-s} \Gamma(s) L_-(s) ds
 \end{aligned} \tag{4.11}$$

where the $r_{\epsilon,n}$ are given by $r_{+,0} = 0$ and

$$r_{+,n} = \frac{(-1)^{n+1} i}{(2n-1)!} (2\pi z)^{2n-1} L_+(1-2n) \quad (n \in \mathbb{Z}_{>0}) \tag{4.12}$$

$$r_{-,n} = \frac{(-1)^n}{(2n)!} (2\pi z)^{2n} L_-(-2n) \quad (n \in \mathbb{Z}_{\geq 0}). \tag{4.13}$$

Proof We have

$$\begin{aligned}
 R_\epsilon(z) &= \sum_{n>0} (T(n) + \epsilon T(-n)) e^{2\pi inz} = \sum_{n>0} \frac{T(n) + \epsilon T(-n)}{2\pi i} \int_{(A)} (-2\pi inz)^{-s} \Gamma(s) ds \\
 &= \frac{1}{2\pi i} \int_{(A)} e^{\frac{1}{2}i\pi s} (2\pi z)^{-s} \Gamma(s) L_\epsilon(s) ds.
 \end{aligned} \tag{4.14}$$

Now let $\epsilon = +$ and move the line of integration to the left, say to the line $(-\frac{1}{2} - 2M)$ for any positive integer M . The poles of the integrand in (4.14), namely $e^{\frac{1}{2}i\pi s} (2\pi z)^{-s} \Gamma(s) L_+(s)$, are simple poles at $s \in -1 + 2\mathbb{Z}_{\leq 0}$. For $n \in \mathbb{Z}_{>0}$ we have

$$\text{Res}_{s=1-2n} \left(e^{\frac{1}{2}i\pi s} (2\pi z)^{-s} \Gamma(s) L_+(s) \right) = \frac{(-1)^{n+1} i}{(2n-1)!} (2\pi z)^{2n-1} L_+(1-2n) = r_{+,n}.$$

One can use the functional equation of L_+ to rewrite this expression of $r_{+,n}$. The same thing can be said for $\epsilon = -$. In this case, the poles of the integrand in (4.14), namely $e^{\frac{1}{2}i\pi s} (2\pi z)^{-s} \Gamma(s) L_-(s)$, are simple poles at $s \in 2\mathbb{Z}_{\leq 0}$. For $n \in \mathbb{Z}_{\geq 0}$ we have

$$\text{Res}_{s=-2n} \left(e^{\frac{1}{2}i\pi s} (2\pi z)^{-s} \Gamma(s) L_-(s) \right) = \frac{(-1)^n}{(2n)!} (2\pi z)^{2n} L_-(-2n) = r_{-,n}.$$

One can use the functional equation of L_- to rewrite this expression of $r_{-,n}$. Since $r_{+,0} = 0$, we see that

$$R_\epsilon(z) = \sum_{n=0}^M r_{\epsilon,n} + \frac{1}{2\pi i} \int_{(-\frac{1}{2}-2M)} e^{\frac{1}{2}i\pi s} (2\pi z)^{-s} \Gamma(s) L_\epsilon(s) ds. \tag{4.15}$$

Hence

$$R(z) = R_+(z) + R_-(z) = \sum_{n=0}^M (r_{+,n} + r_{-,n}) + \frac{1}{2\pi i} \int_{(-\frac{1}{2}-2M)} e^{\frac{1}{2}i\pi s} (2\pi z)^{-s} \Gamma(s) L(s) ds.$$

Now let us look at $\frac{1}{z}R_\epsilon\left(-\frac{1}{z}\right)$ for $\epsilon \in \{+, -\}$. We have

$$\begin{aligned} \frac{1}{z}R_\epsilon\left(-\frac{1}{z}\right) &= \frac{1}{z} \frac{1}{2\pi i} \int_{(A)} e^{\frac{1}{2}i\pi s} \left(\frac{-2\pi}{z}\right)^{-s} \Gamma(s) L_\epsilon(s) ds \\ &= \frac{1}{2\pi i} \int_{(A)} e^{-\frac{1}{2}i\pi s} (2\pi)^{-s} z^{s-1} \Gamma(s) L_\epsilon(s) ds \\ &= \frac{1}{2\pi i} \int_{(1-A)} e^{-\frac{1}{2}i\pi(1-s)} (2\pi)^{-(1-s)} z^{-s} \Gamma(1-s) L_\epsilon(1-s) ds. \end{aligned}$$

One applies the functional equation of $\Lambda_\epsilon(s)$ (Theorem 4.6) to see that

$$\begin{aligned} \frac{1}{z}R_+\left(-\frac{1}{z}\right) &= \frac{1}{2\pi i} \int_{(1-A)} (-i) \cot \frac{\pi s}{2} 1152^{s-\frac{1}{2}} e^{\frac{1}{2}i\pi s} (2\pi z)^{-s} \Gamma(s) L_+(s) ds \\ \frac{1}{z}R_-\left(-\frac{1}{z}\right) &= \frac{1}{2\pi i} \int_{(1-A)} (-i) \tan \frac{\pi s}{2} 1152^{s-\frac{1}{2}} e^{\frac{1}{2}i\pi s} (2\pi z)^{-s} \Gamma(s) L_-(s) ds. \end{aligned}$$

Since the current integrands do not have poles when $\text{Re } s \leq 0$, we are free to move the line of integration to the left and deduce that

$$\frac{1}{z}R_+\left(-\frac{1}{z}\right) = \frac{1}{2\pi i} \int_{(-\frac{1}{2}-2M)} (-i) \cot \frac{\pi s}{2} 1152^{s-\frac{1}{2}} e^{\frac{1}{2}i\pi s} (2\pi z)^{-s} \Gamma(s) L_+(s) ds \tag{4.16}$$

$$\frac{1}{z}R_-\left(-\frac{1}{z}\right) = \frac{1}{2\pi i} \int_{(-\frac{1}{2}-2M)} (-i) \tan \frac{\pi s}{2} 1152^{s-\frac{1}{2}} e^{\frac{1}{2}i\pi s} (2\pi z)^{-s} \Gamma(s) L_-(s) ds. \tag{4.17}$$

We now note two identities:

$$e^{\frac{1}{2}i\pi s} + ie^{\frac{1}{2}i\pi s} \cot \frac{\pi s}{2} = \frac{i}{\sin \frac{\pi s}{2}} \tag{4.18}$$

$$e^{\frac{1}{2}i\pi s} - ie^{\frac{1}{2}i\pi s} \tan \frac{\pi s}{2} = \frac{1}{\cos \frac{\pi s}{2}}. \tag{4.19}$$

We add (4.15) and (4.16) (with z replaced by $1152z$) and use (4.18) to conclude (4.10). We add (4.15) and (4.17) (with z replaced by $1152z$) and use (4.19) to conclude (4.11). \square

Proceeding in the same fashion as Corollaries 4.4 and 4.5, with the aid of Theorem 4.6 we arrive at:

Corollary 4.8 *Let M be a positive integer. One has*

$$R_+ \left(\frac{it}{2\pi} \right) = \sum_{n=1}^M \frac{-L_+(1-2n)}{(2n-1)!} t^{2n-1} + O(t^{2M+\frac{1}{2}}) - \frac{2\pi i}{1152^{\frac{1}{2}}t} R_+ \left(\frac{2\pi i}{1152t} \right) \quad (\operatorname{Re} t > 0, |t| \rightarrow 0)$$

$$R_- \left(\frac{it}{2\pi} \right) = \sum_{n=0}^M \frac{L_-(-2n)}{(2n)!} t^{2n-1} + O(t^{2M+\frac{1}{2}}) + \frac{2\pi i}{1152^{\frac{1}{2}}t} R_- \left(\frac{2\pi i}{1152t} \right) \quad (\operatorname{Re} t > 0, |t| \rightarrow 0).$$

We are now ready to see that each of the functions $q^{\frac{1}{24}}\sigma(q) = R\left(\frac{it}{48\pi}\right)$ and $q^{-\frac{1}{24}}\sigma^*(q) = R^*\left(\frac{it}{48\pi}\right)$ satisfies HYPO 1 and HYPO 3. Recall $R = \frac{1}{2}(R_+ + R_-)$ and $R^* = \frac{1}{2}(R_+ - R_-)$.

Corollary 4.9 (1) *Let M be a positive integer. Let $q = e^{2\pi iz} = e^{-t}$. As $|t| \rightarrow 0$ in the bounded cone $|\arg t| < \frac{\pi}{2} - \delta$ and $|\operatorname{Im} t| \leq \pi$, one has*

$$e^{-\frac{t}{24}}\sigma(e^{-t}) = \frac{1}{2}L_-(0) + \sum_{n=1}^M \left(-\frac{L_+(1-2n)}{(2n-1)!2} \left(\frac{t}{24}\right)^{2n-1} + \frac{L_-(-2n)}{(2n)!2} \left(\frac{t}{24}\right)^{2n} \right) + O(t^{2M+\frac{1}{2}}),$$

$$e^{\frac{t}{24}}\sigma^*(e^{-t}) = -\frac{1}{2}L_-(0) + \sum_{n=1}^M \left(-\frac{L_+(1-2n)}{(2n-1)!2} \left(\frac{t}{24}\right)^{2n-1} - \frac{L_-(-2n)}{(2n)!2} \left(\frac{t}{24}\right)^{2n} \right) + O(t^{2M+\frac{1}{2}}).$$

(2) *As $|t| \rightarrow 0$ in the bounded cone $\frac{\pi}{2} - \delta \leq |\arg t| < \frac{\pi}{2}$ and $|\operatorname{Im} t| \leq \pi$, there is a positive real constant C (independent of t) for which*

$$e^{-\frac{t}{24}}\sigma(e^{-t}) \ll_{\delta} t^{-C} \quad \text{and} \quad e^{\frac{t}{24}}\sigma^*(e^{-t}) \ll_{\delta} t^{-C}.$$

Let us simplify the notation of Corollary 4.9 by the following

Definition 4.10 Let the coefficients σ_n and σ_n^* be given by

$$e^{-\frac{t}{24}}\sigma(e^{-t}) =: \sum_{n=0}^{\infty} \frac{\sigma_n}{n!} \left(\frac{t}{24}\right)^n \quad (\operatorname{Re} t > 0)$$

$$e^{\frac{t}{24}}\sigma^*(e^{-t}) =: \sum_{n=0}^{\infty} \frac{\sigma_n^*}{n!} \left(\frac{t}{24}\right)^n \quad (\operatorname{Re} t > 0).$$

4.4 The Kontsevich–Zagier function $F(q)$

Kontsevich defined the series

$$F(q) = 1 + \sum_{n=1}^{\infty} (q)_n.$$

This series does not converge in any open subset of \mathbb{C} , but it makes sense when q is a root of unity. Zagier [82] investigated this series in great details. An interesting aspect of his work is a “strange identity”

$$\sum_{n=0}^{\infty} (q)_n = 1 + \sum_{n=1}^{\infty} (1-q) \cdots (1-q^n) = -\frac{1}{2} \sum_{n=1}^{\infty} n \left(\frac{12}{n}\right) q^{\frac{n^2-1}{24}}. \tag{4.20}$$

To quote Zagier: “The meaning of the equality is that the function on the left agrees at roots of unity with the radial limit of the function on the right, and similarly for the derivatives of all orders. Equation (4.20), of which

$$e^{-\frac{t}{24}} \sum_{n=0}^{\infty} (1 - e^{-t}) \cdots (1 - e^{-nt}) = \sum_{n=0}^{\infty} \frac{T_n}{n!} \left(\frac{t}{24}\right)^n$$

is a consequence, is related to the Dedekind η -function and the theory of periods of modular forms.”

To be precise, Zagier defined for $|q| < 1$ the following functions

$$F_1(q) = \sum_{n=1}^{\infty} n(q)_{n-1} q^n = \lim_{N \rightarrow \infty} \sum_{n=1}^N n(q)_{n-1} q^n$$

$$F_2(q) = \sum_{n=0}^{\infty} [(q)_n - (q)_{\infty}]$$

$$H(q) = \sum_{n=1}^{\infty} n \left(\frac{12}{n}\right) q^{\frac{n^2-1}{24}} \quad (|q| < 1)$$

$$= 1 - 5q - 7q^2 + 11q^5 + 13q^7 - \dots$$

$$E(q) = \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} = \sum_{n=1}^{\infty} d(n)q^n = q + 2q^2 + 2q^3 + 3q^4 + \dots$$

Theorem 4.11 (Zagier [82])

- (1) One has $F_1(q) = F_2(q) = -\frac{1}{2}H(q) + \left(\frac{1}{2} - E(q)\right) (q)_{\infty}$ as an identity in $\mathbb{Z} \left[\frac{1}{2}\right] [[q]]$. Note however that $F_1 = F_2 \in \mathbb{Z}[[q]]$.
- (2) As $t \in \mathbb{R}_+, t \rightarrow 0$ one has

$$e^{-\frac{t}{24}} H(e^{-t}) = \sum_{n=1}^{\infty} n \left(\frac{12}{n}\right) e^{-\frac{n^2 t}{24}} \sim \sum_{n=0}^{\infty} \gamma_n t^n$$

where $\gamma_n = \frac{(-1)^n}{24^n n!} L\left(-2n - 1, \left(\frac{12}{\cdot}\right)\right)$.

One can then use the same arguments as Sect. 4.3 to deduce the following:

Corollary 4.12 (1) Let M be a positive integer. As $|t| \rightarrow 0$ in the bounded cone $|\arg t| < \frac{\pi}{2} - \delta$ one has

$$e^{-\frac{t}{24}} F_1(e^{-t}) = e^{-\frac{t}{24}} F_2(e^{-t}) = -2 \sum_{n=1}^M \gamma_n t^n + O(t^{M+1})$$

where $\gamma_n = \frac{(-1)^n}{24^n n!} L\left(-2n - 1, \left(\frac{12}{\cdot}\right)\right)$ and $-2\gamma_n = \frac{T_n}{24^n n!}$.

- (2) As $|t| \rightarrow 0$ in the bounded cone $\frac{\pi}{2} - \delta \leq |\arg t| < \frac{\pi}{2}$ there is a positive real constant C (independent of t) for which

$$e^{-\frac{t}{24}} F_1(e^{-t}) = e^{-\frac{t}{24}} F_2(e^{-t}) \ll_{\delta} t^{-C}.$$

5 Proofs of combinatorial probability theorems

In this section we prove the main theorems stated in Sect. 1.

Proof of Theorem 1.1 Recall that $p_{10}(N)$ is the number of partitions of N for which $sm(\lambda) \equiv 1 \pmod{2}$. The first identity of Theorem 1.7 expresses the generating function $\sum_{N \geq 0} p_{10}(N)q^N$ as the product of a modular form and a function with quantum modular behavior.

We show the additional q -identity

$$\sum_{n=0}^{\infty} (-1)^n (q)_n = \frac{1}{2} \left(1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (q)_n \right) = \frac{1}{2} \sigma(q). \tag{5.1}$$

Consider the basic hypergeometric q -series

$$F(a, b; t) = F(a, b; t : q) := \sum_{n=0}^{\infty} \frac{(aq; q)_n}{(bq; q)_n} t^n.$$

When $|t| = 1$, but $t \neq 1$ and $|q| < 1$ the series converges via the recurrence

$$F(a, b; t) = \frac{1-b}{1-t} + \frac{b-atq}{1-t} F(a, b; tq), \tag{5.2}$$

which appears as (2.4) in [42]. Applying (5.2) with $t = -1, a = 1, b = 0$, we obtain (5.1).

Using the notation of Sects. 1.5 and 3, it follows from (1.5) that

$$p_{10}(N) = V(N) := \frac{1}{2\pi i} \int_C \frac{L\xi}{q^{N+1}} dq$$

where $L(q) = \frac{1}{(2\pi)^{\frac{1}{2}}} q^{\frac{1}{24}} \sum_{n=0}^{\infty} (-1)^n ((q)_n - (q)_{\infty})$ and $\xi(q) = \frac{(2\pi)^{\frac{1}{2}}}{\eta(q)}$. Corollary 4.9 shows that $L(e^{-t})$ verifies HYPO 1 and HYPO 3, whereas Theorem 4.1 shows that $\xi(e^{-t})$ verifies HYPO 2 and HYPO 4. In the notation of Sect. 1.5, we have

$$B = 0, \quad \alpha_s = \frac{1}{(2\pi)^{\frac{1}{2}} 2} \cdot \frac{\sigma_s}{s! 24^s},$$

$$\beta = \frac{1}{2}, \quad c^2 = \frac{\pi^2}{6}, \quad \gamma = 4\pi^2.$$

Proposition 1.8 then yields

$$V(N) = e^{2\pi\sqrt{\frac{N}{6}}} N^{-1} \left(\sum_{r=0}^{R-1} A_r N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right) \right)$$

where $A_r = \sum_{s=0}^r \alpha_s w_{s,r-s}$ and

$$w_{s,r} = \frac{\left(\frac{-1}{4}\right)^r \left(\frac{\pi}{\sqrt{6}}\right)^{s+1-r}}{2\pi^{\frac{1}{2}}} \left(s + \frac{3}{2}, r\right).$$

This concludes Theorem 1.1. □

Proof of Theorem 1.2 It follows from (1.6) that

$$e_{sd}(N) = V(N) := \frac{1}{2\pi i} \int_C \frac{-L\xi}{q^{N+1}} dq - d(N)$$

where $L(q) = \frac{1}{(2\pi)^{\frac{1}{2}}} q^{\frac{1}{24}} F(q)$ and $\xi(q) = \frac{(2\pi)^{\frac{1}{2}}}{\eta(q)}$. Corollary 4.9 shows that $L(e^{-t})$ verifies HYPO 1 and HYPO 3, whereas Theorem 4.1 shows that $\xi(e^{-t})$ verifies HYPO 2 and HYPO 4. In the notation of Sect. 1.5, we have

$$B = 0, \quad \alpha_s = \frac{-\gamma_s}{(2\pi)^{\frac{1}{2}} 2} = \frac{T_s}{(2\pi)^{\frac{1}{2}} s! 24^s},$$

$$\beta = \frac{1}{2}, \quad c^2 = \frac{\pi^2}{12}, \quad \gamma = 2\pi^2.$$

Proposition 1.8 then yields

$$V(N) = e^{\pi\sqrt{\frac{N}{3}}} N^{-1} \left(\sum_{r=0}^{R-1} B_r N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right) \right)$$

where $B_r = \sum_{s=0}^r \alpha_s w_{s,r-s}$ and

$$w_{s,r-s} = \frac{1}{2\sqrt{\pi}} (-2K')^{s-r} \left(\frac{K'}{2}\right)^{s+1/2} (s+1, r-s).$$

This concludes Theorem 1.2. □

Proof of Theorem 1.3 By definition we have

$$\sum_{\lambda} (-1)^{\ell(\lambda)+1} \text{sm}(\lambda) q^{|\lambda|} = \sum_{n=0}^{\infty} nq^n (q^{n+1}; q)_{\infty}. \tag{5.3}$$

Theorem 1.3 is now an immediate consequence of (5.3) and an identity which was proved in [62, Section 6.3]:

$$\sum_{n=1}^{\infty} nq^n (q^{n+1}; q)_{\infty} = \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}.$$

See also (1.8) and its proof in Sect. 2. □

Proof of Theorem 1.4 Using the notation of Sect. 3, it follows from (1.7) that

$$e_{sm}(N) = V(N) := \frac{1}{2\pi i} \int_C \frac{L\xi}{q^{N+1}} dq$$

where $L(q) = \frac{1}{(2\pi)^{\frac{1}{2}}} q^{\frac{1}{24}} F(q)$ and $\xi(q) = \frac{(2\pi)^{\frac{1}{2}}}{n(q)}$. Corollary 4.12 shows that $L(e^{-t})$ verifies HYPO 1 and HYPO 3, whereas Theorem 4.1 shows that $\xi(e^{-t})$ verifies HYPO 2 and HYPO 4. In the notation of Sect. 1.5, we have

$$B = 0, \quad \alpha_s = \frac{-\gamma_s}{(2\pi)^{\frac{1}{2}} 2} = \frac{T_s}{(2\pi)^{\frac{1}{2}} s! 24^s},$$

$$\beta = \frac{1}{2}, \quad c^2 = \frac{\pi^2}{6}, \quad \gamma = 4\pi^2.$$

Proposition 1.8 then yields

$$V(N) = e^{2\pi\sqrt{\frac{N}{6}}} N^{-1} \left(\sum_{r=0}^{R-1} D_r N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right) \right)$$

where $D_r = \sum_{s=0}^r \alpha_s w_{s,r-s}$ and

$$w_{s,r-s} = \frac{1}{2\sqrt{\pi}} (-2K)^{s-r} \left(\frac{K}{2}\right)^{s+1} \left(s + \frac{3}{2}, r-s\right)$$

This concludes Theorem 1.4. □

Proof of Theorem 1.5 Using the notation of Sect. 3, it follows from (1.8) that

$$e_{1g}(N) = V(N) := \frac{1}{2\pi i} \int_C \frac{L\xi}{q^{N+1}} dq;$$

where $L(q) = \frac{1}{(2\pi)^{\frac{1}{2}}} q^{\frac{1}{24}} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}$ and $\xi(q) = \frac{(2\pi)^{\frac{1}{2}}}{\eta(q)}$. Corollary 4.5 shows that $L(e^{-t})$ verifies HYPO 1 and HYPO 3, whereas Theorem 4.1 shows that $\xi(e^{-t})$ verifies HYPO 2 and HYPO 4. Moreover, we write

$$L(e^{-t}) = \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{t}{24}} \left(-\frac{\log t}{t} + \frac{\gamma}{t} + \frac{1}{4} + \sum_{n=1}^{\infty} \frac{B_{2n}^2}{(2n)!(2n)} t^{2n-1} \right)$$

and splits into two parts: (a) $-\frac{\log t}{t}$, (b) $\frac{\gamma}{t} - \frac{1}{4} + \sum_{n=1}^{\infty} \frac{B_{2n}^2}{(2n)!(2n)} t^{2n-1}$. In the notation of Sect. 1.5, we now have a combination of the logarithmic-type case with half-integer order [part (a)] and of the polynomial-type case [part (b)]. Proposition 1.10 applies to part (a), whereas Proposition 1.8 applies to part (b). For example, the parameters in the logarithmic-type case are

$$\begin{aligned} B &= 1, & \alpha_0 &= \frac{-1}{(2\pi)^{\frac{1}{2}}}, \\ \beta &= \frac{1}{2}, & c^2 &= \frac{\pi^2}{6}, & \gamma &= 4\pi^2. \end{aligned}$$

Applying Propositions 1.8 and 1.10, we conclude Theorem 1.5. □

Proof of Theorem 1.6 Using the notation of Sect. 3, it follows from (1.8) that

$$e_{1d}(N) = V(N) := \frac{1}{2\pi i} \int_C \frac{L\xi}{q^{N+1}} dq$$

where $L(q) = 2^{-\frac{1}{2}} q^{-\frac{1}{24}} \left(-\frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}\right)$ and $\xi(q) = 2^{\frac{1}{2}} q^{\frac{1}{24}} (-q)_{\infty}$. Corollary 4.5 shows that $L(e^{-t})$ verifies HYPO 1 and HYPO 3, whereas Theorem 4.1 shows that $\xi(e^{-t})$ verifies HYPO 2 and HYPO 4. Moreover, we write

$$L(e^{-t}) = 2^{-\frac{1}{2}} e^{\frac{t}{24}} \left(-\frac{\log t}{t} + \frac{\gamma}{t} - \frac{1}{4} + \sum_{n=1}^{\infty} \frac{B_{2n}^2}{(2n)!(2n)} t^{2n-1} \right)$$

and splits into two parts: (a) $-\frac{\log t}{t}$, (b) $\frac{\gamma}{t} - \frac{1}{4} + \sum_{n=1}^{\infty} \frac{B_{2n}^2}{(2n)!(2n)} t^{2n-1}$. In the notation of Sect. 1.5, we now have a combination of the logarithmic-type case with integer order [part (a)] and of the polynomial-type case [part (b)]. Proposition 1.9 applies to part (a), whereas

Proposition 1.8 applies to part (b). For example, the parameters in the logarithmic-type case are

$$B = 1, \quad \beta = 0, \quad c^2 = \frac{\pi^2}{12}, \quad \gamma = 2\pi^2.$$

Applying Propositions 1.8 and 1.9, we conclude Theorem 1.6. \square

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