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A geometric perspective on p-adic properties of mock modular forms

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Abstract

Bringmann et al. (Trans Am Math Soc 364(5):2393–2410, 2012) showed how to 'regularize' mock modular forms by a certain linear combination of the Eichler integral of their shadows in order to obtain p-adic modular forms in the sense of Serre. In this paper, we give a new proof of a refined form of their results (for good primes p) by employing the geometric theory of harmonic Maass forms developed by Candelori (Math Ann 360(1–2):489–517, 2014) and the theory of overconvergent modular forms due to Katz and Coleman. In particular, our main results imply that the p-adic modular forms in Bringmann et al. (2012) are overconvergent.

Mathematics Subject Classification: 11F33, 11F23

1 Background

Over the past decade, there has been a renewed interest in Ramanujan's *mock modular forms* and related objects, such as *harmonic (weak) Maass forms*, whose Fourier coefficients have been found in many instances to encode interesting arithmetic data, similarly as in the classical theory of modular forms. In this paper, we introduce a new perspective on the p-adic properties of Fourier coefficients of mock modular forms, based on the algebro-geometric theory of p-adic modular forms due Katz [12] and Coleman [8]. Such p-adic properties were originally discovered by Guerzhoy–Kent–Ono [11] and Bringmann–Guerzhoy–Kane [1], but we believe that our methods offer a most natural approach to such results.

In order to state our results precisely, let $\tau = u + iv \in \mathfrak{h}$ be the variable in Poincaré's upper-half plane, with $u, v \in \mathbb{R}$, let $\Gamma_0(N)$ be the standard congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N, and let χ be a Dirichlet character modulo N. Denote by $\mathcal{H}_k(\Gamma_0(N), \chi)$ the space of harmonic Maass forms on $\Gamma_0(N)$ of integral weight k and character χ (as defined in [1, §2]). Any harmonic Maass form F has a decomposition

$$F = F^+ + F^-$$

into a holomorphic part F^+ with poles supported at the cusps and a nonholomorphic part F^-. After Zwegers' work [15] (see also [14] for an influential overview), the function $F^+ : \mathfrak{h} \to \mathbb{C}$ is called a *mock modular form*; in general, it does not transform like a modular form, but (as first discovered by Ramanujan) the properties of its Fourier coefficients resemble those of a classical modular form.

As shown in [3], harmonic Maass forms map into classical modular forms via differential operators. Denote by $M_k^!(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$) the space of weakly holomorphic modular forms (resp. cusp forms) of weight k, level N, and character χ. If for any $w \in \mathbb{Z}$, we let

$$\xi_w := 2iv^w \overline{\frac{\partial}{\partial \overline{\tau}}}, \tag{1}$$

then $f := \xi_{2-k}(F) = \xi_{2-k}(F^-)$ is a cusp form in $S_k(\Gamma_0(N), \chi)$ for all $F \in \mathcal{H}_{2-k}(\Gamma_0(N), \overline{\chi})$. We say that f is the *shadow* of F, and a fundamental question in the subject is to relate the coefficients of a mock modular form F^+ to the coefficients of its shadow.

However, with the differential operator (1) having an infinite-dimensional kernel, to obtain results in this direction it becomes necessary to work with a refined notion of harmonic Maass forms lifting a given f. For any congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, let $S_k(\Gamma, K)$ (resp. $M_k^!(\Gamma, K)$) be the space of cusp forms (weakly homomorphic modular forms) of weight k and level Γ whose q-expansion coefficients all lie in $K \subseteq \mathbb{C}$.

Definition 1.1 A harmonic Maass form $F \in \mathcal{H}_{2-k}(\Gamma_1(N))$ is *good* for $f \in S_k(\Gamma_1(N), K)$ if:

(i) The principal parts of F at all cusps are defined over K.
(ii) We have $\xi_{2-k}(F) = f/\|f\|^2$, where $\|f\|$ is the Petersson norm of f.

Suppose that $f \in S_k(\Gamma_1(N), K)$ is a (normalized) newform defined over K, let F be a harmonic Maass form that is good for f, and write

$$F^+ = \sum_{n \gg -\infty} c^+(n) q^n$$

for the holomorphic part of F. Let $E_f = \sum_{n=1}^{\infty} n^{1-k} a_n q^n$ be the so-called Eichler integral of f, so that $D^{k-1}(E_f) = f$ for the differential operator D^{k-1} acting as $(qd/dq)^{k-1}$ on q-expansions. It is shown in [11] (and in Theorem 4.1 below by different methods) that for any $\alpha \in \mathbb{C}$ such that $\alpha - c^+(1) \in K$, the coefficients of

$$\mathcal{F}_\alpha := F^+ - \alpha E_f = \sum_{n \gg -\infty} c_\alpha(n) q^n$$

also lie in K. In particular, this applies of course to $\alpha = c^+(1)$.

Now fix a prime $p \nmid N$, and a choice of complex and p-adic embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, and let v_p be the resulting p-adic valuation on $\overline{\mathbb{Q}}$ normalized so that $v_p(p) = 1$. Thus, for any value of α in the set

$$c^+(1) + \mathbb{C}_p := \{c^+(1) + \gamma \ : \ \gamma \in \mathbb{C}_p\}, \tag{2}$$

the q-expansion of $\mathcal{F}_\alpha$ lies in $\mathbb{C}_p[[q]][q^{-1}]$, and it becomes meaningful to ask about the p-adic properties of its coefficients; in particular, whether the resulting q-expansion corresponds to a p-adic modular form. In general, the coefficients $c_\alpha(n)$ of $\mathcal{F}_\alpha$ will have unbounded p-adic valuation (see, e.g., [1, p. 2396]), but the following special case of our main result shows that, for a specific value of α, a certain regularization of $\mathcal{F}_\alpha$ indeed gives rise to a p-adic modular form.

For the statement, let β and β' be the roots of the Hecke polynomial of f at p:

$$T^2 - a_p T + \chi(p) p^{k-1} = (T - \beta)(T - \beta'),$$

ordered so that $v_p(\beta) \leqslant v_p(\beta')$. Let V be the operator acting as $q \mapsto q^p$ on q-expansions.

Theorem 1.2 *With the above notations and hypotheses, suppose* $v_p(\beta) < v_p(\beta')$ *and* $v_p(\beta') < k-1$, *and set* $\mathcal{F}_\alpha^* := \mathcal{F}_\alpha - p^{1-k}\beta' V(\mathcal{F}_\alpha)$. *Then, among all values* $\alpha \in c^+(1) + \mathbb{C}_p$, *the value*

$$\alpha = c^+(1) + (\beta - \beta') \lim_{w\to+\infty} \frac{c_{c^+(1)}(p^w)}{\beta^{w+1}}$$

is the unique one such that $\mathcal{F}_\alpha^*$ *is an overconvergent modular form of weight* $2-k$.

We refer the reader to Definition 3.1 for the precise notion of overconvergent modular forms to which Theorem 1.2 applies, but suffice it to say that they bear a relation to Coleman's overconvergent modular forms [8] analogous to that of p-adic modular forms in the sense of [1] to Serre's p-adic modular forms [13]. In particular, our results in Sect. 5 (of which Theorem 1.2 is a special case) yield a new proof of a refined form of the main results obtained by Bringmann–Guerzhoy–Kane in [1], showing that the p-adic modular forms constructed in *loc.cit.* are overconvergent.

We conclude this Introduction by briefly mentioning some key ideas behind our proof of Theorem 1.2. Let f_β and $f_{\beta'}$ be the p-stabilizations of f, which are modular forms of level Np that are eigenvectors for the U-operator with eigenvalues β and β', respectively. In Theorem 4.3 we show that, for all but one value of α, the p-stabilized shadow f_β can be recovered from an iterated application of U on $D^{k-1}(\mathcal{F}_\alpha)$; the exceptional value of α yields the precise value in Theorem 1.2. The forms f_β and $f_{\beta'}$ define classes in the f-isotypical component of a certain parabolic cohomology group, and in Proposition 3.4 we show that under the assumptions of Theorem 1.2 they form a basis for this space. Writing the class of $D^{k-1}(\mathcal{F}_\alpha)$ in terms of this basis, our proof of Theorem 4.3 then follows from an analysis of the action of U on cohomology.

2 Harmonic Maass forms: the geometric point of view

We begin by briefly recalling the geometric interpretation of harmonic Maass forms given in [4]. For $N > 4$, consider the moduli functor $\mathscr{M}_1(N)$ of generalized elliptic curves with a point of order N, which is represented by a smooth and proper scheme over $\mathbb{Z}[1/N]$. Let $\mathcal{E}^{\mathrm{gen}} \to \mathscr{M}_1(N)$ be the universal generalized elliptic curve, and let $\underline{\omega}$ be its relative dualizing sheaf. Let $X := \mathscr{M}_1(N) \times_{\mathbb{Z}[1/N]} \mathbb{Q}$ and $Y := X \smallsetminus C$, where C is the cuspidal subscheme, whose ideal sheaf we denote by $\mathcal{I}_C$. For any extension $K/\mathbb{Q}$, we denote by X_K, Y_K the base-change to K.

We have well-known canonical isomorphisms

$$M_k^!(\Gamma_1(N), K) \simeq H^0(Y_K, \underline{\omega}^k), \quad S_k(\Gamma_1(N), K) \simeq H^0(X_K, \underline{\omega}^k \otimes \mathcal{I}_C),$$

where a modular form f of weight k is identified with the differential $f(dq/q)^k$. Let $\pi : \mathcal{E} \to Y$ be the universal elliptic curve with $\Gamma_1(N)$-level structure. The relative de Rham cohomology of $\pi : \mathcal{E} \to Y$ canonically extends to a rank two vector bundle $\mathcal{H}^1_{\mathrm{dR}}$ over X. Let

$$\mathcal{H}_r := \mathrm{Sym}^r(\mathcal{H}^1_{\mathrm{dR}}).$$

The Gauss–Manin connection of $\pi : \mathcal{E} \to Y$ extends to a connection with logarithmic poles $\nabla : \mathcal{H}^1_{\mathrm{dR}} \to \mathcal{H}^1_{\mathrm{dR}} \otimes \Omega^1_X(\log C)$ over X, and we let

$$\nabla_r : \mathcal{H}_r \longrightarrow \mathcal{H}_r \otimes \Omega^1_X(\log C)$$

denote its r-th symmetric power. Define

$$\mathbb{H}^1_{\mathrm{par}}(X, \mathcal{H}_r) := \mathbb{H}^1(X, \mathcal{H}_r \otimes \mathcal{I}_C \xrightarrow{\nabla_r} \mathcal{H}_r \otimes \Omega^1_X), \tag{3}$$

where $\mathbb{H}^\bullet$ denotes hypercohomology. The formation of $\mathbb{H}^1_{\rm par}(X, \mathcal{H}_r)$ is compatible with base-change under field extensions $K/\mathbb{Q}$, and over $\mathbb{C}$ it is canonically isomorphic to the parabolic cohomology group attached to the space of cusp forms of weight $r+2$ and level $\Gamma_1(N)$. In particular, by the Shimura isomorphism (see, e.g., [9, Thm. 2.10], [2]) $\mathbb{H}^1_{\rm par}(X_{\mathbb{C}}, \mathcal{H}_r)$ is canonically isomorphic to the direct sum of $S_{r+2}(\Gamma_1(N))$ and its complex conjugate.

More generally, the following second description of $\mathbb{H}^1_{\rm par}(X, \mathcal{H}_r)$ in terms of modular forms will play an important role here. Recall that for all $k \geqslant 2$ there is a differential operator

$$D^{k-1} : M^!_{2-k}(\Gamma_1(N)) \longrightarrow M^!_k(\Gamma_1(N))$$

acting on q-expansion as $(qd/dq)^{k-1}$. In particular, D^{k-1} preserves fields of definition.

Theorem 2.1 ([4, Thm. 6]) *Let K be a subfield of $\mathbb{C}$ and let $S^!_k(\Gamma_1(N), K)$ be the subspace of those modular forms in $M^!_k(\Gamma_1(N), K)$ with vanishing constant coefficient in their q-expansions at the cusps. Then, for all $k \geqslant 2$ there is a canonical isomorphism:*

$$\mathbb{H}^1_{\rm par}(X_K, \mathcal{H}_{k-2}) \simeq \frac{S^!_k(\Gamma_1(N), K)}{D^{k-1}M^!_{2-k}(\Gamma_1(N), K)}.$$

The spaces $\mathbb{H}^1_{\rm par}(X_K, \mathcal{H}_{k-2})$ are endowed with an action of the Hecke operators T_ℓ for all primes $\ell \nmid N$, and if $f \in S_k(\Gamma_1(N), K)$ is a newform, we let

$$M_{\rm dR}(f) := \mathbb{H}^1_{\rm par}(X_K, \mathcal{H}_{k-2})^f$$

denote the f-isotypical component for this action. Note that this is a 2-dimensional K-vector space. This can be seen by extending scalars to $\mathbb{C}$ and then noting that the Shimura isomorphism is compatible under the action of Hecke operators, thus $M_{\rm dR}(f) \otimes_K \mathbb{C} \simeq \mathbb{C}[f] \oplus \mathbb{C}[\bar{f}]$.

Now let $[\phi]$ be a class in $M_{\rm dR}(f)$ represented by an element $\phi \in S^!_k(\Gamma_1(N), K)$ using Theorem 2.1. Extending scalars to $\mathbb{C}$, we may write

$$[\phi] = s_1[f] + s_2[\bar{f}], \tag{4}$$

for some $s_1, s_2 \in \mathbb{C}$. Let C^∞_Y (resp. $\mathcal{A}^1_Y$) be the sheaf of smooth functions (resp. smooth differential forms) on $Y_{\mathbb{C}}$. The differential $\phi - s_1 f - s_2\bar{f}$ is smooth over $Y_{\mathbb{C}}$, and it defines a class in

$$\mathbb{H}^1(\mathcal{H}_{k-2} \otimes C^\infty_Y \xrightarrow{\nabla_{k-2}} \mathcal{H}_{k-2} \otimes \mathcal{A}^1_Y) \simeq \frac{H^0(Y_{\mathbb{C}}, \mathcal{H}_{k-2} \otimes \mathcal{A}^1_Y)}{\nabla_{k-2}H^0(Y_{\mathbb{C}}, \mathcal{H}_{k-2} \otimes C^\infty_Y)}$$

which is trivial by construction. Therefore, there exists a smooth $\mathcal{H}_{k-2}$-valued section $\mathbf{F}$ such that

$$\nabla_{k-2}(\mathbf{F}) = \phi - s_1 f - s_2\bar{f}.$$

The vector bundle $\mathcal{H}_{k-2}$ decomposes into line bundles as

$$\mathcal{H}_{k-2} \simeq \underline{\omega}^{2-k} \oplus \underline{\omega}^{4-k} \oplus \cdots \oplus \underline{\omega}^{k-2},$$

and we let $F \in \underline{\omega}^{2-k}$ be the projection of $\mathbf{F}$ to the first factor. With this construction, it is shown in [4, Prop. 4] that F is a harmonic Maass form of weight $2-k$ satisfying

$$D^{k-1}(F^+) = \phi - s_1 f, \quad \frac{2iv^{2-k}}{(-4\pi)^{k-1}}\frac{\partial}{\partial\bar{\tau}}(F^-) = s_2\bar{f}.$$

Carrying out the above construction of F with a class $[\phi]$ normalized so that $\langle f, \phi\rangle = 1$ under the cup product, one then finds that the constant s_2 in (4) is given

$$s_2 = 1/\langle f, \bar{f}\rangle = 1/(-4\pi)^{k-1}\|f\|^2,$$

which shows that $\xi_{2-k}(F) = f/\|f\|^2$ and F is good for f in the sense of Definition 1.1.

3 Overconvergent modular forms

Let $p \geqslant 5$ be a prime and let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. We fix a valuation v_p on $\mathbb{C}_p$ such that $v_p(p) = 1$ and an absolute value $|\cdot|$ on $\mathbb{C}_p$ compatible with v_p. Let K_p be a complete discretely valued subfield of $\mathbb{C}_p$ and let R_p be its ring of integers. Suppose $(p, N) = 1$, and let $\mathscr{X} := \mathscr{M}_1(N) \times_{\mathbb{Z}[1/N]} R_p$ be the base-change to R_p. Let $E_{p-1} \in H^0(\mathscr{X} \times_{R_p} K_p, \underline{\omega}^{p-1})$ be the global section given by the Eisenstein series of weight $p-1$ and level 1, normalized so that its constant coefficient is 1. As in [8, §1], for any $\epsilon \in |R_p|$ there are rigid analytic spaces $X_{(\epsilon)}$ characterized by

$$X_{(\epsilon)}^{\mathrm{cl}} = \{x \in (\mathscr{X} \times_{R_p} K_p)^{\mathrm{cl}} : |E_{p-1}(x)| > \epsilon\},$$

where the superscript 'cl' denotes the set of closed points. In the terminology of [6], the spaces $X_{(\epsilon)}$ for $0 < \epsilon < 1$ are *wide-open neighborhoods* of the *ordinary locus* X^{ord} of X, which is the rigid analytic space characterized by

$$(X^{\mathrm{ord}})^{\mathrm{cl}} = \{x \in (\mathscr{X} \times_{R_p} K_p)^{\mathrm{cl}} : |E_{p-1}(x)| \geqslant 1\}.$$

Since $|E_{p-1}(c)| = 1$ for all $c \in C$, we have $C \subseteq X_{(\epsilon)}$ for all $\epsilon \in |R_p|$, and we let

$$Y^{\mathrm{ord}} := X^{\mathrm{ord}} \setminus C, \quad Y_{(\epsilon)} := X_{(\epsilon)} \setminus C$$

be the rigid analytic spaces obtained by removing the cusps. The invertible sheaves $\underline{\omega}^k$ restrict to rigid analytic line bundles on these spaces denoted in the same manner.

Definition 3.1 An *overconvergent modular form* of integral weight k is a rigid analytic section of $\underline{\omega}^k$ on $Y_{(\epsilon)}$ for some $\epsilon < 1$.

Remark 3.2 As shown by Katz [12], sections of $\underline{\omega}^k$ over X^{ord} are the same as Serre's p-adic modular forms [13] of integral weight k, and therefore elements in $H^0(Y^{\mathrm{ord}}, \underline{\omega}^k)$ correspond to p-adic modular forms in the sense considered in [1]. As explained in [*loc.cit.*, p. 2394], the latter give rise to Serre's p-adic modular forms upon multiplication by an appropriate power of the modular discriminant $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$, and the same argument shows that overconvergent modular forms in the sense of Definition 3.1 give rise to overconvergent modular forms in the sense of Coleman [8].

For any wide-open neighborhood W of X^{ord}, set $W^\circ := W \setminus C$ and define

$$\mathbb{H}^1(W^\circ, \mathcal{H}_r) := \mathbb{H}^1(W^\circ, \mathcal{H}_r \xrightarrow{\nabla_r} \mathcal{H}_r \otimes \Omega^1_X) \simeq \frac{H^0(W^\circ, \mathcal{H}_r \otimes \Omega^1_X)}{\nabla_r H^0(W^\circ, \mathcal{H}_r)},$$

where the isomorphism follows from the fact that $H^q(W^\circ, \mathcal{H}) = 0$ for $q > 0$ and any coherent sheaf $\mathcal{H}$ on W°. The next two results will play an important role in the proofs of our main results.

Theorem 3.3 (Coleman) *For every $r \geqslant 0$, there is linear map*

$$\theta^{r+1} : H^0(W^\circ, \underline{\omega}^{-r}) \longrightarrow H^0(W^\circ, \underline{\omega}^{r+2})$$

whose action on q-expansions is $(qd/dq)^{r+1}$, *and the natural injection*

$$H^0(W^\circ, \underline{\omega}^{r+2}) \simeq H^0(W^\circ, \underline{\omega}^r \otimes \Omega^1_X) \hookrightarrow H^0(W^\circ, \mathcal{H}_r \otimes \Omega^1_X)$$

induces an isomorphism

$$\mathbb{H}^1(W^\circ, \mathcal{H}_r) \simeq \frac{H^0(W^\circ, \underline{\omega}^{r+2})}{\theta^{r+1} H^0(W^\circ, \underline{\omega}^{-r})}.$$

Proof See [8, Prop. 4.3] for the construction of θ^{r+1} and [*loc.cit.*, Thm. 5.4] for the last isomorphism. □

Consider now the wide-open neighborhoods of X^{ord} given by $W_1 := X_{(p^{-p/p+1})}$ and $W_2 := X_{(p^{-1/p+1})} \subseteq W_1$, and let

$$U\colon H^0(W_2, \underline{\omega}^k) \longrightarrow H^0(W_1, \underline{\omega}^k), \quad V\colon H^0(W_1, \underline{\omega}^k) \longrightarrow H^0(W_2, \underline{\omega}^k)$$

be the operators defined in [8, §§2, 3] and whose action on q-expansions is given by the usual formulas

$$U\Big(\sum_n a_n q^n\Big) = \sum_n a_{pn} q^n, \quad V\Big(\sum_n a_n q^n\Big) = \sum_n a_n q^{pn}.$$

Let $f = \sum_{n=1}^\infty a_n q^n \in S_k(\Gamma_0(N), \chi)$ be a newform defined over a number field K and with T_p-eigenvalue a_p. This is a section of $\underline{\omega}^k$ defined over X, and thus by restriction it gives a section of $\underline{\omega}^k$ over W_2 as well. The relation $T_p = U + \chi(p)p^{k-1}V$ trivially implies that

$$a_p f = U(f) + \chi(p)p^{k-1}V(f) \in H^0(W_2, \underline{\omega}^k),$$

from which it follows easily that the *p-stabilizations*

$$f_\beta := f - \beta' V(f) \quad f_{\beta'} := f - \beta V(f) \tag{5}$$

are U-eigenvectors with eigenvalues β and β', respectively. After replacing K by a quadratic extension if necessary, we assume from now on that both β and β' lie in K.

Let K_p be the completion of K at the prime above p induced by our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, and set $M_{\mathrm{dR},p}(f) := M_{\mathrm{dR}}(f) \otimes_K K_p$. For any wide-open neighborhood W of X^{ord}, the natural restriction

$$\mathbb{H}^1_{\mathrm{par}}(X_{K_p}, \mathcal{H}_{k-2}) \longrightarrow \mathbb{H}^1(W^\circ, \mathcal{H}_{k-2}) \tag{6}$$

is injective. (See [6, Thm. 4.2] for the case $k = 2$ and [7, Prop. 10.3] for higher weights.) The image of this map can be described in terms of p-adic residues, and as a result for any newform f as above, the classes $[f_\beta], [f_{\beta'}] \in \mathbb{H}^1(W_2^\circ, \mathcal{H}_{k-2})$ naturally lie in $\mathbb{H}^1_{\mathrm{par}}(X_{K_p}, \mathcal{H}_{k-2})$. In fact, similarly as $\mathbb{H}^1_{\mathrm{par}}(X_{K_p}, \mathcal{H}_{k-2})$, the spaces $\mathbb{H}^1(W^\circ, \mathcal{H}_{k-2})$ are endowed with an action of the Hecke operators T_ℓ for $\ell \nmid Np$ (see [7, §8]), and the restriction map (6) is equivariant for these actions. Therefore, the classes $[f_\beta], [f_{\beta'}]$ naturally lie in $M_{\mathrm{dR},p}(f)$.

Proposition 3.4 *Let* $f = \sum_{n=1}^\infty a_n q^n \in S_k(\Gamma_0(N), \chi)$ *be a newform of weight* $k \geqslant 2$, *and let* β *and* β' *be the roots of* $T^2 - a_p T + \chi(p)p^{k-1}$, *ordered so that* $v_p(\beta) \leqslant v_p(\beta')$. *Assume that the following two conditions hold:*

(i) $\beta \neq \beta'$.
(ii) $f_{\beta'} \notin \mathrm{im}(\theta^{k-1})$.

Then $\{[f], [V(f)]\}$ *is a basis for* $M_{\mathrm{dR},p}(f)$.

Proof Since $M_{\mathrm{dR},p}(f) := M_{\mathrm{dR}}(f) \otimes_K K_p$ is 2-dimensional (see remark after Thm. 2.1), it suffices to show that the classes $[f]$ and $[V(f)]$ are linearly independent. Since clearly $v_p(\beta) < k-1$, by [8, Lem. 6.3] we have $[f_\beta] \neq 0$. Thus, by conditions (i) and (ii) the classes $[f_\beta]$ and $[f_{\beta'}]$ are linearly independent. On the other hand, from the definitions (5) we see that

$$\begin{bmatrix} f \\ V(f) \end{bmatrix} = \frac{1}{\beta - \beta'} \begin{bmatrix} \beta & 1 \\ -\beta' & -1 \end{bmatrix} \begin{bmatrix} f_\beta \\ f_{\beta'} \end{bmatrix},$$

and since $\det \left[\begin{smallmatrix} \beta & 1 \\ -\beta' & -1 \end{smallmatrix}\right] = \beta' - \beta \neq 0$, the result follows. □

Remark 3.5 By results of Coleman–Edixhoven [5], condition (i) in Proposition 3.4 holds if $k = 2$, and for $k > 2$ it is a consequence of the semi-simplicity of crystalline Frobenius, which remains an open conjecture. On the other hand, by [8, Prop. 7.1] condition (ii) fails if f has CM by an imaginary quadratic field in which p splits, and the 'p-adic variational Hodge conjecture' of Emerton–Mazur (see [10]) predicts that these are the *only* cases where it fails.

4 Recovering the shadow

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalized newform and let F be a harmonic Maass form which is good for f in the sense Definition 1.1. By the construction in Sect. 2, we may assume that F satisfies

$$D^{k-1}(F) = D^{k-1}(F^+) = \phi - s_1 f \tag{7}$$

for some $\phi \in S_k^!(\Gamma_1(N), K)$ and $s_1 \in \mathbb{C}$.

In [11], Guerzhoy, Kent, and Ono showed that one of the p-stabilizations of f can be recovered p-adically from an iterated application of U to a certain 'regularization' of $D^{k-1}(F^+)$. In this section, we give a new proof of this result using the p-adic techniques developed above. We begin by giving a new proof of [*loc.cit.*, Thm. 1.1].

Theorem 4.1 *Let* $\alpha \in \mathbb{C}$ *be such that* $\alpha - c^+(1) \in K$*. Then, the coefficients of*

$$\mathcal{F}_\alpha := F^+ - \alpha E_f := \sum_{n \gg -\infty} c^+(n) q^n - \alpha \sum_{n=1}^{\infty} a_n n^{1-k} q^n$$

are all in K.

Proof Write $\phi = \sum_{n \gg -\infty} d(n) q^n$, with $d(n) \in K$. By (7), we have the formula

$$c^+(n) = \left(\frac{d(n) - s_1 a_n}{n^{k-1}} \right) \tag{8}$$

where $a_n := 0$ for $n \leqslant 0$. The result is thus clear for $n \leqslant 0$. Now let $n \geqslant 1$, and write $\alpha = c^+(1) + \gamma$ with $\gamma \in K$, or equivalently, $\alpha = d(1) - s_1 + \gamma$. Using (8), an immediate calculation reveals that the coefficient of q^n in $\mathcal{F}_\alpha$ is given by $(d(n) - d(1) - \gamma) n^{1-k}$. □

Since one can always take $\alpha = c^+(1)$ in Theorem 4.1, the coefficients of $\mathcal{F}_{c^+(1)}$ are all in K. Writing

$$D^{k-1}(\mathcal{F}_{c^+(1)}) = \sum_{n \gg -\infty} c_{c^+(1)}(n) q^n,$$

we may thus view the coefficients $c_{c^+(1)}(n)$ inside $\mathbb{C}_p$ via our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Our next result is a special case of [11, Thm. 1.2(i)], but the ideas in the proof of the general case (see Theorem 4.3 below) already appear here.

Theorem 4.2 *Assume that $v_p(\beta) < v_p(\beta')$ and that $f_{\beta'} \notin \mathrm{im}(\theta^{k-1})$. Then*

$$\lim_{w \to +\infty} \frac{U^w D^{k-1}(\mathcal{F}_{c^+(1)})}{c_{c^+(1)}(p^w)} = f_\beta.$$

Proof First note that by Eqs. (7) and (8), we have

$$D^{k-1}(\mathcal{F}_{c^+(1)}) = \phi - d(1) f,$$

which is a weakly holomorphic cusp form of weight k with q-expansion coefficients in K, hence defining a class in $M_{\mathrm{dR}}(f)$ (see Theorem 2.1). Our assumptions clearly imply conditions (i) and (ii) of Proposition 3.4, and so (as shown in the proof) the K_p-vector space $M_{\mathrm{dR},p}(f)$ has a basis $\{[f_\beta], [f_{\beta'}]\}$ of eigenvectors for U. In particular, we can write

$$[D^{k-1}(\mathcal{F}_{c^+(1)})] = t_1[f_\beta] + t_2[f_{\beta'}]$$

for some constants $t_1, t_2 \in K_p$. By restriction, the differential $D^{k-1}(\mathcal{F}_{c^+(1)}) - t_1 f_\beta - t_2 f_{\beta'}$ defines a class in $\mathbb{H}^1(W_2^\circ, \mathcal{H}_{k-2}) \simeq H^0(W_2^\circ, \underline{\omega}^k)/\theta^{k-1} H^0(W_2^\circ, \underline{\omega}^{2-k})$ which is trivial by construction. Thus, we may write

$$D^{k-1}(\mathcal{F}_{c^+(1)}) = t_1 f_\beta + t_2 f_{\beta'} + \theta^{k-1} h$$

for some $h \in H^0(W_2^\circ, \underline{\omega}^{2-k})$. Applying U to both sides of the equation gives

$$UD^{k-1}(\mathcal{F}_{c^+(1)}) = t_1 \beta f_\beta + t_2 \beta' f_{\beta'} + U(\theta^{k-1} h);$$

and more generally, for any power $w \geqslant 1$, we obtain

$$U^w D^{k-1}(\mathcal{F}_{c^+(1)}) = t_1 \beta^w f_\beta + t_2 \beta'^w f_{\beta'} + U^w(\theta^{k-1} h). \tag{9}$$

Dividing by β^w, we get

$$\beta^{-w} U^w D^{k-1}(\mathcal{F}_{c^+(1)}) = t_1 f_\beta + t_2 \left(\frac{\beta'}{\beta}\right)^w f_{\beta'} + \beta^{-w} U^w(\theta^{k-1} h)$$

and taking the limit as $w \to +\infty$ we arrive at

$$\lim_{w \to +\infty} \beta^{-w} U^w D^{k-1}(\mathcal{F}_{c^+(1)}) = t_1 f_\beta.$$

Here we used the hypothesis $v_p(\beta'/\beta) > 0$, and that fact that (since the coefficients of h have bounded denominators) the differential $U^w(\theta^{k-1} h)$ has coefficients with arbitrarily high valuation as $w \to +\infty$.

To determine the value of t_1, consider the coefficient of q^{p^w} in (9), which is given by

$$\begin{aligned} c_{c^+(1)}(p^w) = a_{p^w}(D^{k-1}(\mathcal{F}_{c^+(1)})) &= a_1(U^w D^{k-1}(\mathcal{F}_{c^+(1)})) \\ &= t_1 \beta^w + t_2 \beta'^w + O(p^{w(k-1)}), \end{aligned}$$

where we let $a_n(g)$ denote the n-th Fourier coefficients in a q-expansion g, and we used the fact that both f_β and $f_{\beta'}$ are normalized, so that $a_1(f_\beta) = a_1(f_{\beta'}) = 1$. Thus, taking the limit as $w \to +\infty$ we obtain

$$\lim_{w\to+\infty} \beta^{-w} c_{c^+(1)}(p^w) = t_1 \tag{10}$$

which gives the result. □

Now for any α with $\alpha - c^+(1) \in K$, define

$$\mathcal{F}_\alpha := F^+ - \alpha E_f$$

and let $c_\alpha(n)$ denote the n-th coefficient in the expansion

$$D^{k-1}(\mathcal{F}_\alpha) = \sum_{n \gg -\infty} c_\alpha(n) q^n.$$

The following is the content of [11, Thm. 1.2(i)] for primes $p \nmid N$.

Theorem 4.3 *Assume that $v_p(\beta) < v_p(\beta')$ and that $f_{\beta'} \notin \mathrm{im}(\theta^{k-1})$. Then for all but at most one choice of α with $\alpha - c^+(1) \in K$, we have*

$$\lim_{w\to+\infty} \frac{U^w D^{k-1}(\mathcal{F}_\alpha)}{c_\alpha(p^w)} = f_\beta.$$

Proof As in the proof of Theorem 4.2, we can write

$$[D^{k-1}(\mathcal{F}_{c^+(1)})] = t_1[f_\beta] + t_2[f_{\beta'}] \tag{11}$$

in $M_{\mathrm{dR},p}(f)$ with the value of t_1 given by (10). Let $\gamma \in K$ be such that $\alpha = c^+(1) + \gamma$, so that $\mathcal{F}_\alpha = \mathcal{F}_{c^+(1)} - \gamma E_f$ by definition. Noting that

$$f = \frac{\beta f_\beta - \beta' f_{\beta'}}{\beta - \beta'}, \tag{12}$$

and substituting into the expression (11) with $\mathcal{F}_\alpha$ in place of $\mathcal{F}_{c^+(1)}$, we obtain

$$[D^{k-1}(\mathcal{F}_\alpha)] = \left(t_1 - \gamma \frac{\beta}{\beta - \beta'}\right)[f_\beta] + \left(t_2 + \gamma \frac{\beta'}{\beta - \beta'}\right)[f_{\beta'}],$$

and hence we have the equality

$$D^{k-1}(\mathcal{F}_\alpha) = \left(t_1 - \gamma \frac{\beta}{\beta - \beta'}\right) f_\beta + \left(t_2 + \gamma \frac{\beta'}{\beta - \beta'}\right) f_{\beta'} + \theta^{k-1} h \tag{13}$$

as sections in $H^0(W_2^\circ, \underline{\omega}^k)$, for some $h \in H^0(W_2^\circ, \underline{\omega}^{2-k})$. Applying U^w to both sides of this equation and letting $w \to +\infty$, we deduce that

$$\lim_{w\to+\infty} \frac{U^w D^{k-1}(\mathcal{F}_\alpha)}{\beta^w} = \left(t_1 - \gamma \frac{\beta}{\beta - \beta'}\right) f_\beta \tag{14}$$

as in the proof of Theorem 4.2. On the other hand, arguing again as in Theorem 4.2, we find that the p^w-th coefficient of $D^{k-1}(\mathcal{F}_\alpha)$ is given by

$$c_\alpha(p^w) = \left(t_1 - \gamma \frac{\beta}{\beta - \beta'}\right) \beta^w + \left(t_2 + \gamma \frac{\beta'}{\beta - \beta'}\right) \beta'^w + O(p^{w(k-1)}),$$

and hence

$$\left(t_1 - \gamma\frac{\beta}{\beta-\beta'}\right) = \lim_{w\to+\infty}\frac{c_\alpha(p^w)}{\beta^w}. \tag{15}$$

Therefore, *except* in the case where

$$\gamma = \frac{t_1(\beta-\beta')}{\beta} = (\beta-\beta')\lim_{w\to+\infty}\frac{c_{c^+(1)}(p^w)}{\beta^{w+1}}, \tag{16}$$

combining (14) and (15) we recover f_β from $\mathcal{F}_\alpha$ as in the statement the theorem. □

5 Mock modular forms as overconvergent modular forms

We now let α range over the larger set of values (2), and interpret the exceptional value of α in Theorem 4.3 as the only value of α for which the 'regularized' mock modular form

$$\mathcal{F}_\alpha = F^+ - \alpha E_f$$

gives rise to an overconvergent modular form (see Definition 3.1) upon p-stabilization. Recall that we let β and β' be the roots of the p-th Hecke polynomial of f, ordered so that $v_p(\beta) \leqslant v_p(\beta')$.

Definition 5.1 For any $\alpha \in c^+(1) + \mathbb{C}_p$, define

$$\mathcal{F}^*_\alpha := \mathcal{F}_\alpha - p^{1-k}\beta'\mathcal{F}_\alpha|V$$

and write $D^{k-1}(\mathcal{F}^*_\alpha) = \sum_{n\gg-\infty} c^*_\alpha(n)q^n$.

Our first result shows that, similarly as in Theorem 4.3 for $\mathcal{F}_\alpha$, the p-stabilization f_β of the shadow of F^+ can be recovered p-adically from $\mathcal{F}^*_\alpha$.

Theorem 5.2 *Assume that $v_p(\beta) < v_p(\beta')$ and that $f_{\beta'} \notin \mathrm{im}(\theta^{k-1})$. Then, for all but at most one choice of $\alpha \in c^+(1) + \mathbb{C}_p$, we have*

$$\lim_{w\to+\infty}\frac{U^w D^{k-1}(\mathcal{F}^*_\alpha)}{c^*_\alpha(p^w)} = f_\beta.$$

Proof Writing $\alpha = c^+(1) + \gamma$ with $\gamma \in \mathbb{C}_p$, an immediate calculation reveals that

$$D^{k-1}(\mathcal{F}^*_\alpha) = D^{k-1}(\mathcal{F}_{c^+(1)})|(1-\beta'V) - \gamma f_\beta. \tag{17}$$

As in the proof of Theorem 4.2, we write

$$[D^{k-1}(\mathcal{F}_{c^+(1)})] = t_1[f_\beta] + t_2[f_{\beta'}]$$

in $M_{\mathrm{dR},p}(f)$ with $t_1 = \lim_{w\to+\infty}\beta^{-w}c_{c^+(1)}(p^w)$. Applying the operator $1-\beta'V$ to this last equality, and noting that $V = U^{-1}$ on cohomology, we obtain

$$[D^{k-1}(\mathcal{F}_{c^+(1)})|(1-\beta'V)] = t_1\frac{(\beta-\beta')}{\beta}[f_\beta],$$

and hence by (17):

$$[D^{k-1}(\mathcal{F}^*_\alpha)] = \left(\frac{t_1(\beta-\beta')}{\beta} - \gamma\right)[f_\beta]. \tag{18}$$

Arguing again as in the proof of Theorem 4.2, we obtain the equalities

$$\lim_{w\to+\infty} \frac{U^w(D^{k-1}(\mathcal{F}_\alpha^*))}{\beta^w} = \left(\frac{t_1(\beta-\beta')}{\beta} - \gamma\right) f_\beta \tag{19}$$

and

$$\frac{t_1(\beta-\beta')}{\beta} - \gamma = \lim_{w\to+\infty} \frac{c_\alpha^*(p^w)}{\beta^w}. \tag{20}$$

Therefore, *except* in the case where

$$\gamma = \frac{t_1(\beta-\beta')}{\beta} = (\beta-\beta') \lim_{w\to+\infty} \frac{c_{c^+(1)}(p^w)}{\beta^{w+1}}, \tag{21}$$

the combination of (14) and (15) recovers f_β from $\mathcal{F}_\alpha^*$ as in the statement. □

Considering the exceptional value of α arising in the proof of Theorem 5.2, we recover a refined form of [1, Thm. 1.1].

Theorem 5.3 *Assume that* $v_p(\beta) < v_p(\beta')$ *and that* $f_{\beta'} \notin \mathrm{im}(\theta^{k-1})$*. Then among all values of* $\alpha \in c^+(1) + \mathbb{C}_p$*, the value*

$$\alpha = c^+(1) + (\beta-\beta') \lim_{w\to+\infty} \frac{c_{c^+(1)}(p^w)}{\beta^{w+1}}$$

is the unique one such that $\mathcal{F}_\alpha^*$ *is an overconvergent modular form of weight* $2-k$*.*

Proof Write $\alpha = c^+(1) + \gamma$ with $\gamma \in \mathbb{C}_p$. Since $[f_\beta] \neq 0 \in M_{\mathrm{dR},p}(f)$ (see the proof of Proposition 3.4), we deduce from (18) and (21) that the class of $D^{k-1}(\mathcal{F}_\alpha^*)$ in $M_{\mathrm{dR},p}(f)$ vanishes only for the value of α in the statement. Since the restriction map

$$\mathbb{H}^1_{\mathrm{par}}(X_{K_p}, \mathcal{H}_{k-2}) \longrightarrow \mathbb{H}^1(W_2^\circ, \mathcal{H}_{k-2}) \simeq \frac{H^0(W_2^\circ, \underline{\omega}^k)}{\theta^{k-1} H^0(W_2^\circ, \underline{\omega}^{2-k})}$$

is injective, the above value of α is also the unique one such that the class of $D^{k-1}(\mathcal{F}_\alpha^*)$ becomes trivial in $\mathbb{H}^1(W_2^\circ, \mathcal{H}_{k-2})$, and hence such that $\mathcal{F}_\alpha^* \in H^0(W_2^\circ, \underline{\omega}^{2-k})$. □

Next we consider a second modification of $\mathcal{F}_\alpha = \sum_{n\gg-\infty} a_{\mathcal{F}_\alpha}(n) q^n$.

Definition 5.4 For any $\delta \in \mathbb{C}_p$, define

$$\mathcal{F}_{\alpha,\delta} := \mathcal{F}_\alpha - \delta(E_f - \beta E_{f|V}).$$

Our next result determines the values of α and δ for which $\mathcal{F}_{\alpha,\delta}$ is an overconvergent modular form, recovering a refined form of [1, Thm 1.2(2)].

Theorem 5.5 *Assume that* $v_p(\beta) < v_p(\beta')$ *and that* $f_{\beta'} \notin \mathrm{im}(\theta^{k-1})$*. Then* $\mathcal{F}_{\alpha,\delta}$ *is an overconvergent modular form for a unique pair* (α, δ)*. In fact,* α *is as in Theorem 5.3, and*

$$\delta = \lim_{w\to+\infty} \frac{a_{\mathcal{F}_\alpha}(p^w) p^{w(k-1)}}{\beta'^w}.$$

Proof With the same notations as in the proof of Theorem 4.3, we can write the equality

$$[D^{k-1}(\mathcal{F}_{\alpha,\delta})] = \left(t_1 - \gamma \frac{\beta'}{\beta - \beta'}\right)[f_\beta] + \left(t_2 + \gamma \frac{\beta}{\beta - \beta'} - \delta\right)[f_{\beta'}] \tag{22}$$

in $M_{\mathrm{dR},p}(f)$. Since we may check the triviality of these classes upon restriction to W_2°, it follows that $\mathcal{F}_{\alpha,\delta}$ is an overconvergent modular form of weight $2-k$ if and only if the class $[D^{k-1}(\mathcal{F}_{\alpha,\delta})]$ vanishes. As in the proof of Proposition 3.4, the classes $[f_\beta]$, $[f_{\beta'}]$ form a basis for $M_{\mathrm{dR},p}(f)$, and hence $\mathcal{F}_{\alpha,\delta}$ is an overconvergent modular form if and only if both coefficients in the right-hand side of (22) vanish. In particular (by the second coefficient), this shows that the value of γ is given by (16), and therefore the necessary value of $\alpha = c^+(1) + \gamma$ is the same as in Theorem 5.3.

To determine the value of δ, we rewrite Eq. (13) for the above value of α (so that the first summand in the right-hand side of that equation vanishes):

$$D^{k-1}(\mathcal{F}_\alpha) = \left(t_2 + \gamma \frac{\beta'}{\beta - \beta'}\right) f_{\beta'} + \theta^{k-1} h.$$

Equating the p^w-th coefficients in this equality we obtain

$$c_\alpha(p^w) = \left(t_2 + \gamma \frac{\beta'}{\beta - \beta'}\right) \beta'^w + O(p^{w(k-1)}),$$

and hence dividing by β'^w and letting $w \to +\infty$ we deduce

$$\lim_{w \to +\infty} \frac{c_\alpha(p^w)}{\beta'^w} = \left(t_2 + \gamma \frac{\beta'}{\beta - \beta'}\right). \tag{23}$$

(Note that the assumption $v_p(\beta') < k-1$ is being used here.) Finally, substituting (23) into (22) we see that the necessary value for δ is given by

$$\delta = \lim_{w \to \infty} \frac{c_\alpha(p^w)}{\beta'^w} = \lim_{w \to \infty} \frac{a_{\mathcal{F}_\alpha}(p^w) p^{w(k-1)}}{\beta'^w},$$

as was to be shown. □

6 The CM case

In this section, we treat the case in which f has CM. This case is of special interest, since then one can choose a good harmonic Maass form F for f as in Section 2 with F^+ having algebraic coefficients.

Thus, assume that $f = \sum_{n=1}^\infty a_n q^n \in S_k(\Gamma_1(N), K)$ has CM by an imaginary quadratic field M of discriminant prime to p, and let $F = F^+ + F^-$ be a good harmonic Maass form attached to f. We also assume (upon enlarging K if necessary) that K contains a primitive m-th root of unity, where $m = N \cdot \mathrm{disc}(M)$. Then, by [3, Thm. 1.3], F^+ has coefficients in K, and so $D^{k-1}(F^+)$ defines a class in $M_{\mathrm{dR}}(f)$.

We first treat the case in which p is inert in M. In this case, $a_p = \beta + \beta' = 0$, and so by the proof of Proposition 3.4, the space $M_{\mathrm{dR},p}(f)$ admits a basis given by the classes $[f_\beta]$ and $[f_{\beta'}]$.

Lemma 6.1 *Assume that p is inert in M, and write $[D^{k-1}(F^+)] = t_1[f_\beta] + t_2[f_{\beta'}]$ with $t_1, t_2 \in K_p$. Then*

$$\lim_{w \to +\infty} \frac{a_{D^{k-1}(F^+)}(p^{2w+1})}{\beta^{2w+1}} = t_1 - t_2.$$

Proof The proof will be obtained by arguments similar to the proof of Theorem 4.2, but some adjustments are necessary due to the fact that condition $\nu_p(\beta) \neq \nu_p(\beta')$ clearly does not hold in this case. Instead, we shall exploit the extra symmetry $\beta' = -\beta$.

Upon restriction to W_2°, we can write

$$D^{k-1}(F^+) = t_1 f_\beta + t_2 f_{\beta'} + \theta^{k-1} h \tag{24}$$

for some $h \in H^0(W_2^\circ, \underline{\omega}^{2-k})$. Taking p^{2w+1}-st coefficients in this identity, we obtain

$$\begin{aligned} a_{D^{k-1}(F^+)}(p^{2w+1}) &= t_1 \beta^{2w+1} + t_2 \beta'^{2w+1} + O(p^{(2w+1)(k-1)}) \\ &= (t_1 - t_2)\beta^{2w+1} + O(p^{(2w+1)(k-1)}), \end{aligned}$$

and hence dividing by β^{2w+1} and letting $w \to +\infty$ the result follows. □

Definition 6.2 For any $\alpha \in \mathbb{C}_p$, define

$$\widetilde{\mathcal{F}}_\alpha := F^+ - \alpha E_{f|V}.$$

Armed with Lemma 6.1, in Corollary 6.4 below we will determine the values of α for which $\widetilde{\mathcal{F}}_\alpha$ is an overconvergent modular form, thus recovering a refined form of [1, Thm. 1.3]. This will be an immediate consequence of the following result.

Theorem 6.3 *Assume that $p \nmid N$ is inert in M, and for any $\widetilde{\alpha} \in \mathbb{C}_p$ define*

$$G_{\widetilde{\alpha}} := F^+ - \widetilde{\alpha}(E_f - \beta E_{f|V}).$$

Then, there exists a unique value of $\widetilde{\alpha}$ such that $G_{\widetilde{\alpha}}$ is an overconvergent modular form of weight $2-k$, and it is given by

$$\widetilde{\alpha} = \lim_{w \to +\infty} \frac{a_{D^{k-1}(F^+)}(p^{2w+1})}{\beta^{2w+1}}.$$

Proof We will deduce this result by first determining the values of α and δ for which the form $\mathcal{F}_{\alpha,\delta}$ of Definition 5.4 is an overconvergent modular form. Note that this case is not covered by Theorem 5.5, since its proof exploits the assumption that $\nu_p(\beta) < \nu_p(\beta')$. However, $[f_\beta]$ and $[f_{\beta'}]$ still form a basis for $M_{\mathrm{dR}}(f)$, and so Eq. (22) for $[D^{k-1}(\mathcal{F}_{\alpha,\delta})]$ applies, yielding (setting $\gamma = \alpha$ by the algebraicity of $c^+(1)$)

$$[D^{k-1}(\mathcal{F}_{\alpha,\delta})] = \left(t_1 - \frac{\alpha}{2}\right)[f_\beta] + \left(t_2 - \frac{\alpha}{2} - \delta\right)[f_{\beta'}]. \tag{25}$$

By Theorem 3.4, the classes $[f]$ and $[V(f)]$ form a basis for $M_{\mathrm{dR}}(f)$, and rewriting (25) in terms of them we arrive at

$$[D^{k-1}(\mathcal{F}_{\alpha,\delta})] = (t_1 + t_2 - \alpha - \delta)[f] + \beta(t_1 - t_2 - \alpha - \delta)[V(f)]. \tag{26}$$

Now, $\mathcal{F}_{\alpha,\delta}$ is an overconvergent modular form if and only if both coefficients in Eq. (26) vanish; in particular, we need to have

$$\alpha + \delta = t_1 - t_2 = \lim_{w \to +\infty} \frac{a_{D^{k-1}(F^+)}(p^{2w+1})}{\beta^{2w+1}}, \tag{27}$$

where we used Lemma 6.1 for the second equality. The necessary vanishing of (26) also forces the vanishing of t_2 and hence from (25) we deduce that $\delta = -\frac{\alpha}{2}$, or equivalently, $\alpha + \delta = \frac{\alpha}{2}$. Finally, noting that

$$\mathcal{F}_{\alpha,\delta} = F^+ - \frac{\alpha}{2}\left(E_f - \beta E_{f|V}\right) = G_{\frac{\alpha}{2}},$$

we conclude from (27) that $G_{\widetilde{\alpha}}$ is an overconvergent modular form of weight $2-k$ if and only if $\widetilde{\alpha}$ is given by the p-adic limit in the statement. □

Corollary 6.4 *Assume that $p \nmid N$ is inert in M. Then, there exists a unique value of α such that $\widetilde{\mathcal{F}}_\alpha$ is an overconvergent modular form of weight $2-k$, and it is given by*

$$\alpha = \lim_{w\to+\infty} \frac{a_{D^{k-1}(F^+)}(p^{2w+1})}{\beta^{2w}}.$$

Proof Comparing the definitions of $\widetilde{\mathcal{F}}_\alpha$ and $G_{\widetilde{\alpha}}$, we see that

$$G_{\widetilde{\alpha}} = \widetilde{\mathcal{F}}_\alpha - \widetilde{\alpha} E_f,$$

with $\alpha = \widetilde{\alpha}\beta$. Since E_f is easily seen to be an overconvergent modular form of weight $2-k$ under our running hypotheses (see [1, Prop. 4.2], which remains true in our case $p \nmid N$), the result follows from Theorem 6.3. □

Finally, we deal with the case in which f has CM by an imaginary quadratic field M in which p splits, characterizing the values of $\alpha \in \mathbb{C}_p$ for which $\mathcal{F}^*_\alpha$ is an overconvergent modular form. As noted in Remark 3.5, the class $[f_{\beta'}]$ vanishes in this case, and so the proofs of Theorems 5.2 and 5.3 break down. However, based on the observation that (using the algebraicity of $c^+(1)$ to set $\alpha = \gamma$)

$$\mathcal{F}^*_\alpha = (F^+ - \alpha E_f)|(1 - p^{1-k}\beta' V) = \mathcal{F}^*_0 - \alpha E_{f_\beta}, \tag{28}$$

we can easily prove the following result (cf. [1, Thm. 1.2]).

Theorem 6.5 *Assume that $p \nmid N$ is split in M. Then, among all values of $\alpha \in \mathbb{C}_p$, the value $\alpha = 0$ is the unique one for which $\mathcal{F}^*_\alpha$ is an overconvergent modular form of weight $2-k$.*

Proof As we have already argued in preceding proofs, $\mathcal{F}^*_\alpha$ is an overconvergent modular form of weight $2-k$ if and only if the class $[D^{k-1}(\mathcal{F}^*_\alpha)]$ vanishes, and from (28) we see that

$$[D^{k-1}(\mathcal{F}^*_\alpha)] = 0 \quad \Longleftrightarrow \quad \alpha[f_\beta] = [D^{k-1}(\mathcal{F}^*_0)].$$

In particular, this shows that $\mathcal{F}^*_\alpha$ is an overconvergent modular form of weight $2-k$ for $\alpha = 0$, and so $[D^{k-1}(\mathcal{F}^*_0)] = 0$. On the other hand, since $[f_\beta] \neq 0$ (see the proof of Proposition 3.4), the above equivalence shows that $[D^{k-1}(\mathcal{F}^*_\alpha)] \neq 0$ for $\alpha \neq 0$, yielding the result. □

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