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A proof of the Thompson moonshine conjecture

Michael J. Griffin[1] and Michael H. Mertens[2*]

*Correspondence:
michael.mertens@emory.edu
[2]Department of Mathematics and Computer Science, Emory University, 400 Dowman Drive, Atlanta, GA 30322, USA
Full list of author information is available at the end of the article

Abstract

In this paper, we prove the existence of an infinite-dimensional graded supermodule for the finite sporadic Thompson group *Th* whose McKay–Thompson series are weakly holomorphic modular forms of weight $\frac{1}{2}$ satisfying properties conjectured by Harvey and Rayhaun.

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1 Introduction and statement of results

One of the greatest accomplishments of twentieth-century mathematics was certainly the classification of finite simple groups. The study of the representation theory of one of these groups, the *Monster group* $\mathbb{M}$, the largest of the 26 sporadic simple groups, revealed an intriguing connection to modular forms: McKay and Thompson [41] were the first to observe that the dimensions of irreducible representations of the Monster group are closely related to Klein's modular invariant

$$J(\tau) = q^{-1} + \sum_{n=1} j_n q^n = q^{-1} + 196\,884q + 21\,493\,760q^2 + 864\,299\,970q^3 + O(q^4),$$

the Hauptmodul for the full modular group. Here and throughout, $\tau = u + iv$, $u, v \in \mathbb{R}$, is a variable living in the complex upper half-plane $\mathfrak{H}$ and $q := e^{2\pi i\tau}$. More precisely, the first irreducible representations of $\mathbb{M}$ have dimensions

$$\chi_1(1) = 1, \quad \chi_2(1) = 196\,883, \quad \chi_3(1) = 21\,296\,876, \quad \chi_4(1) = 842\,609\,326,$$

so that one has (for example)

$$\begin{aligned} j_1 &= \chi_1(1) + \chi_2(1),\\ j_2 &= \chi_1(1) + \chi_2(1) + \chi_3(1),\\ j_3 &= 2\chi_1(1) + 2\chi_2(1) + \chi_3(1) + \chi_4(1). \end{aligned}$$

Thompson further observed that a similar phenomenon occurs when one considers combinations of $\chi_j(g)$ for other $g \in \mathbb{M}$. Based on theses observations, he conjectured in [40] that there should exist an infinite-dimensional graded $\mathbb{M}$-module that reflects these combinations. Conway and Norton [14] made this more precise (the so-called *Monstrous Moonshine* conjecture), conjecturing that for each conjugacy class of $\mathbb{M}$ there is an explicit associated genus 0 subgroup of $\mathrm{SL}_2(\mathbb{R})$ whose normalized Hauptmodul coincides with the so-called *McKay–Thompson series* (see Sect. 2.3 for a definition of this term) of the conjugacy class with respect to the module. An abstract proof of this conjecture (i.e., one whose construction of the module is done only implicitly in terms of the McKay–Thompson series) was announced by Atkin–Fong–Smith [20,36]. Their proof was based on an idea of Thompson. Later, the full conjecture was proven by Borcherds [2] using a vertex operator algebra previously constructed by Frenkel et al. [22].

Conway and Norton also observed in [14] that Monstrous Moonshine would imply Moonshine phenomena for various subgroups of the Monster. Queen [32] computed the Hauptmoduln associated with conjugacy classes of several sporadic groups, among them the Thompson group. Note, however, that the Moonshine phenomenon we prove in this paper is not directly related to this generalized moonshine considered by Queen, but more reminiscent of the following. In 2011, Eguchi et al. [19] observed connections like the ones between the dimensions of irreducible representations of the Monster group and coefficients of the modular function J for the largest Mathieu group M_{24} and a certain weight $\frac{1}{2}$ *mock theta function*. Cheng et al. [12,13] generalized this to Moonshine for groups associated with the 23 Niemeier lattices, the non-isometric even unimodular root lattices in dimension 24, which has become known as the *Umbral Moonshine Conjecture*. Gannon proved the case of Mathieu moonshine in [23], and the full Umbral Moonshine Conjecture was then proved in [18] by Duncan, Ono, and the first author.

In [25], Harvey and Rayhaun conjecture Moonshine for the *Thompson group Th*, a sporadic simple group of order

$$90\,745\,943\,887\,872\,000 = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$$

(see Sect. 2.1 for definitions and notation).

Conjecture 1.1 *There exists a graded Th-supermodule*

$$W = \bigoplus_{\substack{m=-3 \\ m\equiv 0,1 \pmod 4}}^{\infty} W_m,$$

where for $m \geq 0$ the graded component W_m has vanishing odd part if $m \equiv 0 \pmod 4$ and vanishing even part if $m \equiv 1 \pmod 4$, such that for all $g \in Th$ the McKay–Thompson series

$$\mathcal{T}_{[g]}(\tau) := \sum_{\substack{m=-3 \\ m\equiv 0,1 \pmod 4}}^{\infty} \mathrm{strace}_{W_m}(g) q^m$$

is a specifically given weakly holomorphic modular form (see Sect. 2) of weight $\frac{1}{2}$ in the Kohnen plus space.

Here, we prove this conjecture.

Theorem 1.2 *Conjecture 1.1 is true. Moreover, if $g \in Th$ does not lie in the class $12A$ or $12B$ (see Remark 3.3), the McKay–Thompson series $\mathcal{T}_{[g]}(\tau)$ is the unique weakly holomorphic modular form in $M^{!,+}_{\frac{1}{2}}(4|g|, \psi_{[g]})$ (see Sect. 2 for notation) satisfying the following conditions:*

- *its Fourier expansion is of the form* $2q^{-3} + \chi_2(g) + (\chi_4(g) + \chi_5(g))q^4 + O(q^5)$*, where* χ_j *is the* jth *irreducible character of Th as given in Tables 1, 2, 3 and 4, and all its Fourier coefficients are integers.*
- *if* $|g|$ *is odd, then the only other pole of order* $\frac{3}{4}$ *is at the cusp* $\frac{1}{2|g|}$*; otherwise, there is only the pole at* ∞*. It vanishes at all other cusps.*

If $|g| \neq 36$*, it suffices to assume that the Fourier expansion is of the form* $2q^{-3} + \chi_2(g) + O(q^4)$*.*

The proof of Theorem 1.2, like the proofs of the Mathieu and Umbral Moonshine Conjectures, relies on the following idea. For each $n \geq 0$, the function defined by

$$\omega_n\colon Th \to \mathbb{C}, \quad g \mapsto \alpha_{[g]}(n),$$

where we write

$$\mathcal{T}_{[g]}(\tau) := \sum_{n=-3}^{\infty} \alpha_{[g]}(n) q^n$$

for the tentative McKay–Thompson series conjectured by Harvey and Rayhaun, is a complex-valued class function on the Thompson group. Therefore, we need to show that $(-1)^n \omega_n$ is a character of *Th* for every n, which is equivalent to the assertion that

$$(-1)^n \omega_n(g) = \sum_{j=1}^{48} m_j(n) \chi_j(g), \tag{1.1}$$

where $m_1, \ldots, m_{48}$ are nonnegative integers and $\chi_1, \ldots, \chi_{48}$ are the irreducible characters of *Th* as defined in Tables 1, 2, 3 and 4. Using a variant of Brauer's characterization of generalized characters due to Thompson (see, e.g., [36, Theorem 1.1]), one can reduce this to a finite calculation.

The rest of the paper is organized as follows. In Sect. 2, we recall some relevant definitions on supermodules, harmonic Maaß forms, and the construction of the (tentative) McKay–Thompson series in [25]. In Sect. 3, we show that these series are in fact all weakly holomorphic modular forms (instead of harmonic weak Maaß forms) with integer Fourier coefficients and that all the multiplicities m_j in (1.1) are integers. Section 4 is concerned with the proof of the positivity of these multiplicities, which finishes the proof of Theorem 1.2. Finally, in Sect. 5, we give some interesting observations connecting the McKay–Thompson series to replicable functions.

2 Preliminaries and notation

2.1 Supermodules

We begin by introducing the necessary definitions and notations in Conjecture 1.1.

Definition 2.1 A vector space V is called a *superspace*, if it is equipped with a $\mathbb{Z}/2\mathbb{Z}$-grading $V = V^{(0)} \oplus V^{(1)}$, where $V^{(0)}$ is called the *even* and $V^{(1)}$ is called the *odd* part of V. For an endomorphism α of V respecting this grading, i.e., $\alpha(V^{(i)}) \subseteq V^{(i)}$, we define its *supertrace* to be

$$\text{strace}(\alpha) = \text{trace}\left(\alpha|_{V^{(0)}}\right) - \text{trace}\left(\alpha|_{V^{(1)}}\right).$$

Now let G be a finite group and (V, ρ) a representation of G. If the G-module V admits a decomposition into an even and odd part as above which is compatible with the G-action, we call V a *G-supermodule*. For a G-subsupermodule W of V and $g \in G$, we then write

$$\text{strace}_W(g) := \text{strace}\left(\rho|_W(g)\right).$$

Note that strace(g) only depends on the conjugacy class of g, which we denote by $[g]$.

2.2 Harmonic Maaß forms

Harmonic Maaß forms are an important generalization of classical, elliptic modular forms. In the weight 1/2 case, they are intimately related to the *mock theta functions*, a term coined by Ramanujan in his famous 1920 deathbed letter to Hardy. It took until the first decade of the twenty-first century before work by Zwegers [43], Bruinier and Funke [7] and Bringmann and Ono [5,6] established the "right" framework for these enigmatic functions of Ramanujan's, namely that of harmonic Maaß forms. Since then, there have been many applications of harmonic Maaß forms both in various fields of pure mathematics, see for instance [1,4,9,16], among many others, and mathematical physics, especially in regard to quantum black holes and wall crossing [15] as well as Mathieu and Umbral Moonshine [12, 13,18,23]. For a general overview on the subject, we refer the reader to [31,42].

Recall the definition of the congruence subgroup

$$\Gamma_0(N) := \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : N|c\right\}$$

of the full modular group $\text{SL}_2(\mathbb{Z})$.

Definition 2.2 We call a smooth function $f: \mathfrak{H} \to \mathbb{C}$ a *harmonic* (*weak*)[1] *Maaß form* of *weight* $k \in \frac{1}{2}\mathbb{Z}$ of *level* N with *multiplier system* ψ, if the following conditions are satisfied:

(1) We have $f|_k\gamma(\tau) = \psi(\gamma)f(\tau)$ for all $\gamma \in \Gamma_0(N)$ and $\tau \in \mathfrak{H}$, where we define

$$f|_k\gamma(\tau) := \begin{cases} (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) & \text{if } k \in \mathbb{Z}, \\ \left(\left(\frac{c}{d}\right)\varepsilon_d\right)^{2k}\left(\sqrt{c\tau+d}\right)^{-2k} f\left(\frac{a\tau+b}{c\tau+d}\right) & \text{if } k \in \frac{1}{2}+\mathbb{Z}. \end{cases}$$

with

$$\varepsilon_d := \begin{cases} 1 & d \equiv 1 \pmod 4, \\ i & d \equiv 3 \pmod 4. \end{cases}$$

and where we assume $4|N$ if $k \notin \mathbb{Z}$.

(2) The function f is annihilated by the *weight k hyperbolic Laplacian*,

$$\Delta_k f := \left[-v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)\right] f \equiv 0.$$

[1] We usually omit the word "weak" from now on.

(3) There is a polynomial $P(q^{-1})$ such that $f(\tau) - P(e^{-2\pi i\tau}) = O(e^{-cv})$ for some $c > 0$ as $v \to \infty$. Analogous conditions are required at all cusps of $\Gamma_0(N)$.

We denote the space of harmonic Maaß forms of weight k, level N and multiplier ψ is denoted by $H_k(N, \psi)$, where we omit the multiplier if it is trivial.

Remark 2.3 (1) Obviously, the weight k hyperbolic Laplacian annihilates holomorphic functions, so that the space $H_k(N, \psi)$ contains the spaces $S_k(N, \psi)$ of cusp forms (holomorphic modular forms vanishing at all cusps), $M_k(N, \psi)$ of holomorphic modular forms, and $M_k^!(N, \psi)$ of weakly holomorphic modular forms (holomorphic functions on $\mathfrak{H}$ transforming like modular forms with possible poles at cusps).

(2) It should be pointed out that the definition of modular forms resp. harmonic Maaß forms with multiplier is slightly different in [25], where the multiplier is included into the definition of the slash operator $f|_k\gamma$, so that multipliers here are always the inverse of the multipliers there.

It is not hard to see from the definition that harmonic Maaß forms naturally split into a holomorphic part and a non-holomorphic part (see, for example, equations (3.2a) and (3.2b) in [7]).

Lemma 2.4 *Let $f \in H_k(N, \psi)$ be a harmonic Maaß form of weight $k \neq 1$ such that $\psi\left(\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)\right) = 1$. Then there is a canonical splitting*

$$f(\tau) = f^+(\tau) + f^-(\tau), \tag{2.1}$$

where for some $m_0 \in \mathbb{Z}$ we have the Fourier expansions

$$f^+(\tau) := \sum_{n=m_0}^{\infty} c_f^+(n) q^n,$$

and

$$f^-(\tau) := \sum_{n=1}^{\infty} \overline{c_f^-(n)} n^{k-1} \Gamma(1-k; 4\pi n v) q^{-n},$$

where $\Gamma(\alpha; x)$ denotes the usual incomplete Gamma-function.

In the theory of harmonic Maaß forms, there is a very important differential operator that associates a weakly holomorphic modular form to a harmonic Maaß form [7, Proposition 3.2 and Theorem 3.7], often referred to as its *shadow*.[2]

Proposition 2.5 *The operator*

$$\xi_k : H_k(N, \psi) \to S_{2-k}\left(N, \overline{\psi}\right), \; f \mapsto \xi_k f := 2iv^k \overline{\frac{\partial f}{\partial \overline{\tau}}}$$

is well defined and surjective with kernel $M_k^!(N, \nu)$. Moreover, we have that

$$(\xi_k f)(\tau) = -(4\pi)^{1-k} \sum_{n=1}^{\infty} c_f^-(n) q^n$$

[2] In the literature, the shadow is often rather associated with the holomorphic part f^+ of a harmonic Maaß form f rather than to f itself.

and we call this cusp form the shadow of f.

The ξ-operator can also be used to define the *Bruinier–Funke pairing*

$$\{\cdot,\cdot\}\colon S_{2-k}(N,\overline{\psi})\times H_k(N,\psi),\quad (g,f)\mapsto\{g,f\}:=\langle g,\xi_k(f)\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the Petersson inner product on the space of cusp forms.

We will later use the following result [7, Proposition 3.5].

Proposition 2.6 *For $f=f^++f^-$ as in Lemma 2.4 and $g=\sum_{n=1}^{\infty}b_nq^n\in S_{2-k}(N,\overline{\psi})$ such that f grows exponentially only at the cusp ∞ and is bounded at all other cusps of $\Gamma_0(N)$, we have that*

$$\{g,f\}=\sum_{n<0}c_f^+(n)b(-n).$$

If f has poles at other cusps, the pairing is given by summing the corresponding terms using the q-series expansions for f and g at each such cusp.

2.3 Rademacher sums and McKay–Thompson series

Here, we recall a few basic facts about Poincaré series, Rademacher sums, and Rademacher series. For further details, the reader is referred to [10,11,17] and the references therein.

An important way to construct modular forms of a given weight and multiplier is through Poincaré series. If one assumes absolute and locally uniform convergence, then the function

$$P_{\psi,k}^{[\mu]}(\tau):=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)}\overline{\psi}(\gamma)q^{\mu}|_k\gamma,$$

where $\Gamma_\infty=\langle\pm T\rangle$ with $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ denotes the stabilizer of the cusp ∞ in $\Gamma_0(N)$ and $\mu\in\frac{\log(\psi(T))}{2\pi i}+\mathbb{Z}$, transforms like a modular form of weight k with multiplier ψ under the action of $\Gamma_0(N)$ and is holomorphic on $\mathfrak{H}$. In fact it is known that we have absolute and locally uniform convergence for weights $k>2$, and in those cases, $P_{\psi,k}^{[\mu]}$ is a weakly holomorphic modular form, which is holomorphic if $\mu\geq 0$ and cuspidal if $\mu>0$.

For certain groups and multiplier systems, one can obtain conditionally, locally uniformly convergent series, now called *Rademacher sums*, for weights $k\geq 1$, by fixing the order of summation as follows. Let for a positive integer K

$$\Gamma_{K,K^2}(N):=\left\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(N)\ :\ |c|<K\text{ and }|d|<K^2\right\}.$$

One can then define the Rademacher sum

$$R_{\psi,k}^{[\mu]}(\tau)=\lim_{K\to\infty}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_{K,K^2}(N)}\overline{\psi}(\gamma)q^{\mu}|_k\gamma.$$

Rademacher observed [33] that one can in addition regularize the summands individually to obtain convergence for weights $k<1$, see, e.g., equations (2.26) and (2.27) in [11]. He originally applied this to obtain an exact formula for the coefficients of the modular j-function.

In this paper, we especially need to look at Rademacher sums of weights $\frac{1}{2}$ and $\frac{3}{2}$ for $\Gamma_0(4N)$ with multiplier

$$\psi_{N,v,h}(\gamma):=\exp\left(-2\pi iv\frac{cd}{Nh}\right),\tag{2.2}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$ and ν, h are integers with $h \mid \gcd(4N, 96)$. First, we establish convergence of these series.

Proposition 2.7 *For a positive integer N and multiplier $\psi = \psi_{N,\nu,h}$ as in (2.2), the Rademacher sums $R^{[-3]}_{\psi,\frac{1}{2}}(\tau)$ and $R^{[3]}_{\overline{\psi},\frac{3}{2}}$ converge locally uniformly on $\mathfrak{H}$ and therefore define holomorphic functions on $\mathfrak{H}$.*

Proof By following the steps outlined in [10, Section 5] to establish the convergence of weight $\frac{1}{2}$ Rademacher series with a slightly different multiplier system (related to that of the Dedekind eta function) *mutatis mutandis*, we find that the Rademacher sums we are interested in converge, assuming the convergence at $s = \frac{3}{4}$ of the Kloosterman zeta function

$$\sum_{c=1}^{\infty} \frac{K_\psi(-3, n, 4Nc)}{(4Nc)^{2s}}$$

with

$$K_\psi(m, n, c) := \sum_{d \pmod c}^{*} \psi(c, d) \left(\frac{c}{d}\right) \varepsilon_d e\left(\frac{m\overline{d} + nd}{c}\right), \tag{2.3}$$

where the * at the sum indicates that it runs over primitive residue classes modulo c, $\overline{d}$ denotes the multiplicative inverse of d modulo c, and $e(\alpha) := \exp(2\pi i\alpha)$ as usual. We omit the subscript if $\psi = 1$. In order to establish positivity of the multiplicities of irreducible characters in Sect. 4, we will show not only convergence of this series, but even explicit estimates for its value, which will complete the proof. □

Since the Rademacher sum $R^{[-3]}_{\psi,\frac{1}{2}}(\tau)$ is 1-periodic by construction, it has a Fourier expansion, which can (at least formally) be established by standard methods. Projecting this function to the Kohnen plus space then yields the function

$$Z_{N,\psi}(\tau) := q^{-3} + \sum_{\substack{n=0 \\ n \equiv 0,1 \pmod 4}}^{\infty} A_{N,\psi}(n) q^n, \tag{2.4}$$

where $A_{N,\psi}$ is given by

$$\begin{aligned} A_{N,\psi}(0) &:= \frac{\pi\sqrt{3}}{2N^{\frac{3}{2}}}(1-i) \sum_{c=1}^{\infty} (1 + \delta_{\text{odd}}(Nc)) \frac{K_\psi(-3, 0, 4Nc)}{(4Nc)^{\frac{3}{2}}}, \\ A_{N,\psi}(n) &:= \frac{\pi\sqrt{2}}{4N} \left(\frac{3}{n}\right)^{\frac{1}{4}} (1-i) \sum_{c=1}^{\infty} (1 + \delta_{\text{odd}}(Nc)) \frac{K_\psi(-3, n, 4Nc)}{4Nc} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{3n}}{Nc}\right). \end{aligned} \tag{2.5}$$

Here, $I_{\frac{1}{2}}$ denotes the usual modified Bessel function of the first kind of order $\frac{1}{2}$ and

$$\delta_{\text{odd}}(k) := \begin{cases} 1 & k \text{ odd}, \\ 0 & k \text{ even}. \end{cases}$$

For each conjugacy class $[g]$ of the Thompson group Th, we associate integers v_g and h_g (where $h_g|96$) as specified in Table 5 and the character $\psi_{[g]} := \psi_{|g|,v_g,h_g}$, where $|g|$ denotes the order of g in Th, as well as a finite sequence of rational numbers $\kappa_{m,g}$ which are also given in Table 5 and define the function

$$\mathcal{F}_{[g]}(\tau) := 2Z_{|g|,\psi_{[g]}}(\tau) + \sum_{\substack{m>0 \\ m^2|h_g|g|}} \kappa_{m,g}\vartheta(m^2\tau), \tag{2.6}$$

with

$$\vartheta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}.$$

This is going to be the explicitly given weakly holomorphic modular form (see Proposition 3.1) mentioned in Conjecture 1.1, meaning that we have

$$\mathcal{T}_{[g]}(\tau) = \mathcal{F}_{[g]}(\tau)$$

for all conjugacy classes $[g]$ of Th.

We now prove and recall some important facts about Rademacher sums that we shall use later on. As in [10, Propositions 7.1 and 7.2], one sees the following.

Proposition 2.8 *The Rademacher sum $R^{[-3]}_{\psi,\frac{1}{2}}(\tau)$ with ψ as in (2.2) is a mock modular form of weight $\frac{1}{2}$ whose shadow is a cusp form with the conjugate multiplier $\overline{\psi}$, which is a constant multiple of the Rademacher sum $R^{[3]}_{\psi,\frac{3}{2}}$.*

Next, we establish the behavior of Rademacher sums at cusps. Here, we have to take into account that the sums we look at are projected into the Kohnen plus space which might affect the behavior at cusps. For a function $f \in M^!_{k+\frac{1}{2}}(\Gamma_0(4N))$, where k is an integer and N is odd, the projection of f to the plus space is defined by

$$f|\mathrm{pr} = (-1)^{\lfloor\frac{k+1}{2}\rfloor}\frac{1}{3\sqrt{2}}\sum_{\nu=-1}^{2}(f|B\cdot A_\nu) + \frac{1}{3}f,$$

where

$$B = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad A_\nu = \begin{pmatrix} 1 & 0 \\ 4N\nu & 1 \end{pmatrix}.$$

Using this projection operator, one sees that the following is true. The proof is similar in nature to that of Proposition 3 in [28] and is carried out in some detail (for the special case where $N = p$ is an odd prime) in [24, Section 2].

Lemma 2.9 *Let N be odd and $f \in H_{k+\frac{1}{2}}(\Gamma_0(4N))$ for some $k \in \mathbb{N}_0$, such that*

$$f^+(\tau) = q^{-m} + \sum_{n=0}^{\infty} a_n q^n$$

for some $m > 0$ with $-m \equiv 0, (-1)^k \pmod 4$ has a non-vanishing principal part only at the cusp ∞ and is bounded at the other cusps of $\Gamma_0(4N)$. Then, the projection $f|pr$ of f to the plus space has a pole of order m at ∞ and has a pole of order $\frac{m}{4}$ either at the cusp $\frac{1}{N}$ if $m \equiv 0 \pmod 4$ or at the cusp $\frac{1}{2N}$ if $-m \equiv (-1)^k \pmod 4$ and is bounded at all other cusps.

Proof In order to compute the expansion of $f|\mathrm{pr}$ at a given cusp $\mathfrak{a}=\frac{a}{c}$, we compute

$$f|\mathrm{pr}|\sigma_{\mathfrak{a}},$$

where $\sigma_{\mathfrak{a}}=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$. Multiplying out the matrices, we see that this is (up to a constant factor) equal to

$$(-1)^{\lfloor\frac{k+1}{2}\rfloor}\frac{1}{3\sqrt{2}}\sum_{\nu=-1}^{2} f|\begin{pmatrix}(4+4N\nu)a+c & (4+4N\nu)b+d\\ 16N\nu a+4c & 16N\nu b+4d\end{pmatrix}+\frac{1}{3}f|\sigma_{\mathfrak{a}}. \tag{2.7}$$

By assumption, this function can only have a pole at ∞ if the denominator of the fraction $\frac{(4+4N\nu)a+c}{16N\nu a+4c}$ in lowest terms (where we allow the denominator to be 0 which we interpret as ∞) is divisible by $4N$, which is easily seen to imply that $N|c$. Since there are only three inequivalent cusps of $\Gamma_0(4N)$ whose denominator is divisible by N, represented by ∞, $\frac{1}{N}$, $\frac{1}{2N}$, we can restrict ourselves to

$$\sigma_{\frac{1}{N}}=\begin{pmatrix}1 & 0\\ N & 1\end{pmatrix}\quad\text{and}\quad \sigma_{\frac{1}{2N}}=\begin{pmatrix}1 & 0\\ 2N & 1\end{pmatrix}.$$

Plugging first $\sigma_{\frac{1}{N}}$ into (2.7), we find for each $\nu=-1\ldots 2$ that

$$f|B\cdot A\cdot\sigma_{\frac{1}{N}}=f|\begin{pmatrix}4+4N\nu+N & \beta_\nu\\ 16N\nu+4N & \delta_\nu\end{pmatrix}\cdot\begin{pmatrix}1 & \delta_\nu-4\beta_\nu\\ 0 & 16\end{pmatrix}=Cf\left(\frac{\tau+\delta_\nu-4\beta_\nu}{16}\right),$$

where $\beta_\nu,\delta_\nu\in\mathbb{Z}$ such that

$$\det\begin{pmatrix}4+4N\nu+N & \beta_\nu\\ 16N\nu+4N & \delta_\nu\end{pmatrix}=1$$

and $C\in\mathbb{C}$ is a constant that a priori depends on ν, but by working out the corresponding automorphy factors, one sees with easy, elementary methods that it does indeed not. Note that it does, however, depend on N. Furthermore, it is not hard to see that the difference $\delta_\nu-4\beta_\nu$ runs through all residue classes modulo 16 that are congruent to N modulo 4. This implies that in

$$\sum_{\nu=-1}^{2} f\left(\frac{\tau+\delta_\nu-4\beta_\nu}{16}\right)$$

only powers of $q^{\frac{1}{16}}$ survive whose exponent is divisible by 4. Since by assumption we have that $f|\sigma_{\frac{1}{N}}=O(1)$ as $\tau\to\infty$, we therefore see that $f|\mathrm{pr}$ has a pole of order $\frac{m}{4}$ at the cusp $\frac{1}{N}$ if and only if m is divisible by 4.

For the cusp $\frac{1}{2N}$, the argumentation is analogous. One finds that

$$f|B\cdot A\cdot\sigma_{\frac{1}{2N}}=f|\begin{pmatrix}2+2N\nu+N & \beta'_\nu\\ 8N\nu+4N & \delta'_\nu\end{pmatrix}\cdot\begin{pmatrix}2 & \delta_\nu-4\beta_\nu\\ 0 & 8\end{pmatrix}=C'_\nu f\left(\frac{2\tau+\delta_\nu-4\beta_\nu}{8}\right),$$

where $\beta'_\nu,\delta'_\nu,C'_\nu$ have the analogous meaning as β_ν,δ_ν,C above, with the only difference that C'_ν actually does depend on ν. The dependence on ν is so that in the summation only powers of $q^{\frac{1}{4}}$ with exponents $\equiv(-1)^k \pmod 4$ survive, which implies our Lemma. □

For even N, it turns out that the Rademacher series are automatically in the plus space. This follows immediately from the next lemma.

Lemma 2.10 *Let $m, n \in \mathbb{Z}$ such that $m \not\equiv n \pmod 4$ and $c \in \mathbb{N}$ be divisible by 8. Then, we have*

$$K(m, n, c) = 0$$

with $K(m, n, c)$ as defined in (2.3).

Proof We write $c = 2^\ell c'$ with $\ell \geq 3$ and c' odd. By the Chinese Remainder Theorem, one easily sees the following multiplicative property of the Kloosterman sum,

$$K(m, n, c) = K\left(m\overline{c'}, n\overline{c'}, 2^\ell\right) \cdot S\left(m\overline{2^\ell}, n\overline{2^\ell}, c'\right), \tag{2.8}$$

where

$$S_c(m, n, c) := \sum_{d \pmod c}^{*} \left(\frac{c}{d}\right) e\left(\frac{m\overline{d} + nd}{c}\right)$$

is a Salié sum and $\overline{c'}$ denotes the inverse of c' modulo 2^ℓ and $\overline{2^\ell}$ denotes the inverse of 2^ℓ modulo c'.

Therefore, it suffices to show the lemma for $c = 2^\ell$ with $\ell \geq 3$. The case where $\ell = 3$ can be checked directly, so assume $\ell \geq 4$ from now on. In this case, it is straightforward to see that

$$\left(\frac{2^\ell}{d}\right) = \left(\frac{2^\ell}{d + 2^{\ell-1}}\right), \quad \varepsilon_d = \varepsilon_{d+2^{\ell-1}}, \quad \text{and} \quad \overline{d + 2^{\ell-1}} = \overline{d} + 2^{\ell-1}.$$

This yields that for $m \not\equiv n \pmod 2$ we have that

$$\left(\frac{2^\ell}{d + 2^{\ell-1}}\right) \varepsilon_{d+2^{\ell-1}} \cdot e\left(\frac{m(\overline{d + 2^{\ell-1}}) + n(d + 2^{\ell-1})}{2^\ell}\right) = -\left(\frac{2^\ell}{d}\right) \varepsilon_d e\left(\frac{m\overline{d} + nd}{2^\ell}\right)$$

for all odd $d \in \{1, \ldots, 2^{\ell-1} - 1\}$, so that the summands in the Kloosterman sum pair up with opposite signs, making the sum 0 as claimed.

If m and n have the same parity, but are not congruent modulo 4, a similar pairing also works. In this case, we find through similar reasoning that for $\ell \geq 5$ we have

$$\begin{aligned}\left(\frac{2^\ell}{d}\right) \varepsilon_d e\left(\frac{m\overline{d} + nd}{2^\ell}\right) &= \left(\frac{2^\ell}{d + 2^{\ell-1}}\right) \varepsilon_{d+2^{\ell-1}} \cdot e\left(\frac{m(\overline{d + 2^{\ell-1}}) + n(d + 2^{\ell-1})}{2^\ell}\right) \\ &= -\left(\frac{2^\ell}{d + 2^{\ell-2}}\right) \varepsilon_{d+2^{\ell-2}} \cdot e\left(\frac{m(\overline{d + 2^{\ell-2}}) + n(d + 2^{\ell-2})}{2^\ell}\right).\end{aligned}$$

Again, we can pair summands with opposite signs, proving the lemma. □

From the preceding two lemmas, we immediately find that the following is true.

Proposition 2.11 *For any $g \in Th$, the function $Z_{[g],\psi_{[g]}}$ is a mock modular form which has a pole of order 3 at ∞, a pole of order $\frac{3}{4}$ at $\frac{1}{2N}$ if N is odd, and vanishes at all other cusps.*

Proof As described in Appendix E of [10], we see that the Rademacher sums $R^{[-3]}_{\psi_{[g]},\frac{1}{2}}(\tau)$ have only a pole of order 3 at ∞ and grow at most polynomially at all other cusps. By Lemmas 2.9 and 2.10, we see that the poles are as described in the proposition. The vanishing at all remaining cusps follows as in [8, Theorem 3.3]. □

3 Identifying the McKay–Thompson series as modular forms

In this section, we want to establish that the multiplicities of each irreducible character are integers. To this end, we first establish the exact modularity and integrality properties of the conjectured McKay–Thompson series $\mathcal{F}_{[g]}(\tau)$, which are stated without proof in [25].

Proposition 3.1 *For each element g of the Thompson group, the function $Z_{|g|,\psi_{[g]}}(\tau)$ as defined in (2.4) lies in the space $M^{+,!}_{\frac{1}{2}}(4|g|,\psi_{[g]}) \leq M^{+,!}_{\frac{1}{2}}(N_{[g]})$ with $N_{[g]}$ as in Table 5.*

Proof As we know from Proposition 2.8, we have

$$\xi_{\frac{1}{2}} Z_{|g|,\psi_{[g]}} \in S^{+}_{\frac{3}{2}}(4|g|,\overline{\psi_{[g]}}) \leq S^{+}_{\frac{3}{2}}(N_{[g]}).$$

The space $S^{+}_{\frac{3}{2}}(N_{[g]})$ now turns out to be zero-dimensional for

$$[g] \in \{1A, 2A, 3A, 3B, 3C, 4A, 4B, 5A, 6A, 6B, 6C, 7A, 8A, 9A, 9B, 10A, 12A, 12B, 12C, 13A, 18A, 36A, 36B, 36C\},$$

which is directly verifiable using the built-in functions for spaces of modular forms in for example Magma [3]. Furthermore, we have the Bruinier–Funke pairing (see Proposition 2.6) combined with Proposition 2.11 which tell us that

$$\left\{\xi_{\frac{1}{2}} Z_{|g|,\psi_{[g]}}, Z_{|g|,\psi_{[g]}}\right\} \doteq c^{-}_{Z_{|g|,\psi_{[g]}}}(3), \tag{3.1}$$

where the dot above the equal sign indicated an omitted multiplicative (nonzero) constant. More precisely, we can apply Proposition 2.6 to the Rademacher sum $R^{[-3]}_{\psi,\frac{1}{2}}$ and then use the same reasoning as in the proof of Lemma 2.9 to see that projection to the plus space only alters this value by a multiplicative nonzero constant, since the additional pole at the cusp $\frac{1}{2|g|}$ (if $|g|$ is odd) is directly forced by the plus space condition.

From (3.1), we can now deduce, because the Petersson inner product is positive definite on the space of cusp forms, that the shadow of $Z_{|g|,\psi_{[g]}}$ must be 0 if every $f \in S^{+}_{\frac{3}{2}}(N_{[g]})$ is $O(q^4)$. Again, this can be checked using built-in features of Magma, therefore showing the claim for

$$[g] \in \{14A, 19A, 20A, 28A, 31A, 31B\}.$$

For the remaining 18 conjugacy classes, one can use the same arguments as above, but with the refinement that instead of looking at the full space $S^{+}_{\frac{3}{2}}(N_{[g]})$, one looks at the (usually) smaller space $S^{+}_{\frac{3}{2}}(4|g|,\overline{\psi_{[g]}})$. Since computing bases for these spaces is not something that a standard computer algebra system can do without any further work, we describe how to go about doing this. Let $f \in S^{+}_{\frac{3}{2}}(4|g|,\overline{\psi_{[g]}})$ for some conjugacy class $[g]$. Then $f \cdot \vartheta$ is a modular form of weight 2 with the same multiplier and level (respectively, trivial multiplier and level $N_{[g]}$). Using programs[3] written by Rouse and Webb [34], one can verify that the algebra of modular forms of level $N_{[g]}$, so in particular the space $M_2(N_{[g]})$, is generated by eta quotients. One can also compute a generating system consisting of eta

[3] Available at http://users.wfu.edu/rouseja/eta/.

quotients for all remaining $N_{[g]}$ that still need to be considered. Since one can actually compute Fourier expansions of expressions like

$$(f|\gamma)(\tau)$$

for $f(\tau) = \prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$ an eta quotient and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ explicitly, see, e.g., [27, Proposition 2.1] it is straightforward linear algebra to compute a basis of the space $M_2(4|g|, \overline{\psi_{[g]}})$ and from there go down to $S^+_{\frac{3}{2}}(4|g|, \overline{\psi_{[g]}})$. A Magma script computing dimensions and bases of these spaces can be obtained from the second author's homepage. Using this script, we find that $\dim S^+_{\frac{3}{2}}(4|g|, \overline{\psi_{[g]}}) = 0$ for

$$[g] \in \{8B, 9C, 12D, 15A, 15B, 24A, 24B, 24C, 24D, 27A, 27B, 27C\}$$

and for all remaining conjugacy classes $[g]$, we find that every $f \in S^+_{\frac{3}{2}}(4|g|, \overline{\psi_{[g]}})$ is $O(q^4)$. This completes the proof. □

Proposition 3.2 *For each $g \in Th$, the functions $\mathcal{F}_{[g]}(\tau) = \sum_{n=-3}^{\infty} c_{[g]}(n)q^n$ as defined in (2.6) are all weakly holomorphic modular forms of weight $\frac{1}{2}$ for the group $\Gamma_0(N_{[g]})$ in Kohnen's plus space with integer Fourier coefficients at ∞.*

Proof We have established in Proposition 3.1 that the Rademacher series $Z_{|g|,\psi_{[g]}}(\tau)$ are all weakly holomorphic modular forms of weight $\frac{1}{2}$ for $\Gamma_0(N_{[g]})$ in the plus space. The given theta corrections are holomorphic modular forms of the same weight and level, hence so is their sum. Furthermore, theta functions do not have poles, so that all poles of $\mathcal{F}_{[g]}$ come from the Rademacher series which has a pole of order 3 only at the cusps of $\Gamma_0(N_{[g]})$ lying above the cusp ∞ on the modular curve $X_0(4|g|)$. Hence, the function

$$\mathcal{F}_{[g]}(\tau) \cdot G(\tau),$$

where $G(\tau) = q^3 + O(q^4) \in S^+_{2k-\frac{1}{2}}(\Gamma_0(4|g|)$ is a cusp form with integer Fourier coefficients, is a weight $2k$ holomorphic modular form with trivial multiplier under the group $\Gamma_0(N_{[g]})$. Now by the choice of $G(\tau)$, we have that $\mathcal{F}_{[g]}(\tau)$ has integer Fourier coefficients if and only if $\mathcal{F}_{[g]}(\tau) \cdot G(\tau)$ has integer Fourier coefficients, which is the case if and only if the Fourier coefficients of this modular form are integers up to the Sturm bound [37]

$$\frac{k}{6}[\mathrm{SL}_2(\mathbb{Z})\colon \Gamma_0(N_{[g]})].$$

In order to compute the necessary Fourier coefficients exactly without relying on the rather slow convergence of the Fourier coefficients of the Rademacher series, one can construct linear combinations of weight $\frac{1}{2}$ weakly holomorphic eta quotients again using the programs[4] written by Rouse and Webb [34] which have the same principal part at ∞ (and the related cusp $\frac{1}{2N}$) as the Rademacher series (see Proposition 2.11) and the same constant terms as the theta corrections if there are any, wherefore their difference must be a weight $\frac{1}{2}$ holomorphic cusp form for $\Gamma_0(N_{[g]})$ with trivial multiplier, which by the Serre–Stark basis theorem [35] is easily seen to be 0.[5] The largest bound up to which coefficients need to be checked turns out to be 384 for $[g] = 24CD$. □

[4] Available at http://users.wfu.edu/rouseja/eta/.
[5] A list of the eta quotients and linear combinations are available from the second author's homepage.

Remark 3.3 As it turns out, the theta correction

$$f(\tau) = -\vartheta(4\tau) + 3\vartheta(36\tau)$$

$[g] = 12AB$ that Harvey and Rayhaun [25, Table 5] give transforms with a different multiplier than the Rademacher series $Z_{12,\psi_{12AB}}(\tau)$: As one computes directly from the fact that

$$-\vartheta(\tau) + 3\vartheta(9\tau)$$

transforms with the multiplier $\psi_{3,1,3}$ under the group $\Gamma_0(12)$, $f(\tau)$ transforms under the group $\Gamma_0(48)$ with the multiplier $\psi_{12,1,3}$, while $Z_{12,\psi_{12AB}}(\tau)$ transforms with the multiplier $\psi_{12,7,12}$. But since both multipliers become trivial on the group $\Gamma_0(144)$, the proposition remains valid.

We can now establish the uniqueness claim in Theorem 1.2 very easily.

Proposition 3.4 *For $[g] \notin \{12A, 12B, 36A, 36B, 36C\}$, we have that the function $\mathcal{F}_{[g]}(\tau) \in M^{!,+}_{\frac{1}{2}}(4|g|, \psi_{[g]})$ is the unique function satisfying the conditions given in Theorem 1.2:*

- *its Fourier expansion is of the form $2q^{-3} + \chi_2(g) + O(q^4)$ and all its Fourier coefficients are integers.*
- *if $|g|$ is odd, then the only other pole of order $\frac{3}{4}$ is at the cusp $\frac{1}{2|g|}$; otherwise, there is only the pole at ∞. It vanishes at all other cusps.*

For $|g| = 36$, $\mathcal{F}_{[g]}(\tau)$ is uniquely determined by additionally fixing the coefficient of q^4 to be $\chi_4(g) + \chi_5(g)$.

Proof As we have used already, the function $2Z_{|g|,\psi_{[g]}}(\tau)$ has the right behavior at the cusps so that $\mathcal{F}_{[g]}(\tau) - 2Z_{|g|,\psi_{[g]}}(\tau)$ is a holomorphic weight $\frac{1}{2}$ modular form. As it turns out, in all cases but the one where $|g| = 36$, this space is at most two dimensional, which can be seen by the Serre–Stark basis theorem if $\psi_{[g]}$ is trivial or through a computation similar to the one described in the proof of Proposition 3.1 if the multiplier is not trivial. Hence prescribing the constant and first term in the Fourier expansion determines the form uniquely. If $|g| = 36$, the space of weight $\frac{1}{2}$ modular forms turns out to be 3 dimensional, so that fixing one further Fourier coefficient suffices to determine the form uniquely. □

We ultimately want to study the multiplicities of the irreducible characters of *Th*. To this end, we now consider the functions

$$\mathcal{F}_{\chi_j}(\tau) := \frac{1}{|Th|} \sum_{g\in Th} \overline{\chi_j(g)} \mathcal{F}_{[g]} = \sum_{n=-3}^{\infty} m_j(n) q^n$$

with $m_j(n)$ as in (1.1), the generating functions of the multiplicities. We want to show that all those numbers $m_j(n)$ are integers. A natural approach for this would be to view $\mathcal{F}_{\chi_j}$ as a weakly holomorphic modular form of weight $\frac{1}{2}$ and level

$$N_{\chi_j} := \operatorname{lcm}\left\{N_{[g]}: \quad \chi_j(g) \neq 0\right\}$$

and then use a Sturm bound type argument as in the proof of Proposition 3.2. However, these levels turn out to be infeasibly large in most cases. For example, we have that

$$N_{\chi_1} = 2\,778\,572\,160,$$

so one would have to compute at least a few 100 million Fourier coefficients of $\mathcal{F}_{\chi_1}$ to make such an argument work, which is entirely infeasible.

This bound can be reduced substantially, however, by breaking the problem into many smaller problems involving simpler congruences, each of which requires far fewer coefficients to prove.

We proceed by a linear algebra argument. Let $\mathbf{C}$ be the coefficient matrix containing the coefficients of the alleged McKay–Thompson series for each conjugacy class. In theory, we have that $\mathbf{C}$ is a $48 \times \infty$ matrix. In practice, we take $\mathbf{C}$ to be a $48 \times B$ matrix with B large. Let $\mathbf{X}$ be the 48×48 matrix with columns indexed by conjugacy classes of Th and rows indexed by irreducible characters, whose $(\chi_i, [g])$th entry is

$$\mathbf{X}_{(\chi_i,[g])} = \chi_i(g) \cdot \frac{|[g]|}{|Th|}.$$

Using the first Schur orthogonality relation for characters,

$$\sum_{g\in Th} \chi_i(g)\chi_j(g) = \begin{cases} |Th| & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

we have that the rows of the matrix $\mathbf{m} := \mathbf{XC}$ which are indexed by the characters χ_i are exactly the multiplicity values under consideration for the given character.

The matrix $\mathbf{C}$ does not have full rank. Besides the duplicated series (such as $\mathcal{T}_{[12A]} = \mathcal{T}_{[12B]}$), we have additional linear relations given in Appendix B.1. As some of these relations involve the theta functions used as correction terms in the construction, let us define $\mathbf{C}^+$ to be the matrix extending $\mathbf{C}$ to include the coefficients of the theta functions $\vartheta(n^2\tau)$, for $n = 1, 2, 3, 6, 9$. Then there are matrices $\mathbf{N}, \mathbf{N}^*$ of dimensions 48×35 and 35×53, respectively, so that $\mathbf{N}^*\mathbf{C}^+$ has full rank and

$$\mathbf{m} = \mathbf{XC} = \mathbf{XNN}^*\mathbf{C}^+.$$

We construct the matrix $\mathbf{N}^*$ by taking a 53×53 identity matrix indexed by conjugacy classes and removing the rows corresponding to one of each duplicate series and also the highest level conjugacy class (as ordered for instance by Table 5) appearing in each of the linear relations. The matrix N may be constructed starting with a 35×35 identity matrix, adding in columns to reconstitute the removed conjugacy classes and removing the columns corresponding to the theta series.

The rows of $\mathbf{N}^*\mathbf{C}^+$ exhibit additional congruence as listed in Appendix B.2. For each prime p dividing the order of the Thompson group, we construct a matrix $\mathbf{M}_p$ to reduce by these congruences. We start as before with a 35×35 identity matrix. Then for each congruence listed, we replace the row of the matrix with index given by the highest weight conjugacy class appearing in the congruence with a new row constructed to reduce by that congruence. Given the congruence

$$\sum_g a_g \mathcal{T}_{[g]} \equiv 0 \pmod{p^s},$$

where the $a_g \in \mathbb{Z}$, the new row will be given by

$$\sum_g p^{-s} a_g \mathfrak{e}_{[g]},$$

where $\mathfrak{e}_{[g]}$ is the elementary basis element.

For instance, in the case $p = 7$, we have the congruence

$$\mathcal{T}_{[1A]} - \mathcal{T}_{[7A]} \equiv 0 \pmod{7^2}.$$

This tells us that the $[7A]$-th row of M_7 should be

$$7^{-2}\left(\mathfrak{e}_{[7A]} - \mathfrak{e}_{[1A]}\right).$$

Assuming for the moment the validity of these congruences, we have that $\mathbf{M}_p\mathbf{N}^*\mathbf{C}^+$ is an integer matrix. Moreover, by construction $\mathbf{M}_p$ is invertible (depending on ordering, we have that $\mathbf{M}_p$ is lower triangular with non-vanishing main diagonal).

In each case, we have computationally verified that the matrix

$$\mathbf{XNM}_p^{-1}$$

is rational with p-integral entries. Since

$$\mathbf{m} = \left(\mathbf{XNM}_p^{-1}\right) \cdot \left(\mathbf{M}_p\mathbf{N}^*\mathbf{C}^+\right).$$

is the product of two p-integral matrices, we have that every multiplicity must also be p-integral.

The congruences listed in Appendix B.2 were found computationally by reducing the matrix $\left(\mathbf{N}^*\mathbf{C}^+\right) \pmod{p}$ and computing the left kernel. After multiplying by a matrix constructed similar to $\mathbf{M}_p$ above so as to reduce by the congruences found, the process was repeated. The list of congruences given represents a complete list, in the sense that the matrix $\left(\mathbf{M}_p\mathbf{N}^*\mathbf{C}^+\right)$ both is integral and has full rank modulo p.

Many of the congruences can be easily proven using standard trace arguments for spaces of modular forms of level pN to level N. For uniformity, we will instead rely on Sturm's theorem following the argument described above. The worst case falls with any congruence involving the conjugacy class $24CD$. These occur for both primes $p = 2$ and 3. The nature of the congruences, however, does not require us to increase the level beyond the corresponding level $N_{24CD} = 1152$. There is a unique normalized cusp form of weight 19/2 and level 4 in the plus space. This form vanishes to order 3 at the cusp ∞ and to order 3/4 at the cusp 1/2. This is sufficient so that multiplying by this cusp form moves these potential congruences into spaces of holomorphic modular forms of weight 10, level 1152. The Sturm bound for this space falls just shy of 2000 coefficients. This bound could certainly be reduced by more careful analysis, but this is sufficient for our needs. The congruences were observed up to 10,000 coefficients. These computations were completed using Sage mathematical software [39].

Remark 3.5 A similar process can be used in the case of Monstrous Moonshine to prove the integrality of the Monster character multiplicities. This gives an (probably[6]) alternate proof the theorem of Atkin–Fong–Smith [20,36]. As in the case of Thompson moonshine, we have calculated a list of congruences for each prime dividing the order of the Monster, proven by means of Sturm's theorem. This is list complete in the sense that once we have reduced by the congruences for a given prime, the resulting forms have full rank modulo that prime. The Monster congruences may be of independent interest and are available upon request to the authors.

[6] We say probably because the proof of Atkin–Fong–Smith relies on results in Margaret Ashworth's (later Millington) Ph.D. thesis (Oxford University, 1964, advised by A. O. L. Atkin), of which the authors were unable to obtain a copy.

4 Positivity of the multiplicities

To establish the positivity of the multiplicities of the irreducible representations, we follow Gannon's work *mutatis mutandis*. First, we notice that by the first Schur orthogonality relation for characters and the triangle inequality, for each irreducible representation ρ with corresponding character χ of Th we have the estimate

$$\begin{aligned}\text{mult}_{\rho_j}(W_k) &= \sum_{[g]\subseteq Th} \frac{1}{|C(g)|}\text{strace}_{W_k}(g)\overline{\chi(g)} \\ &\geq \frac{|\text{strace}_{W_k}(1)|}{|G|}\chi(1) - \sum_{[g]\neq[1]} \frac{|\text{strace}_{W_k}(g)|}{|C(g)|}|\chi(g)|.\end{aligned}$$

Here, $C(g)$ denotes the centralizer of g in Th and the summation runs over all conjugacy classes of Th. Thus to prove positivity, we show that $|\text{strace}_{W_k}(1)|$ always dominates all the others. To this end, we use the description of the Fourier coefficients of $\mathcal{F}_{[g]}(\tau)$ in terms of Maaß–Poincaré series, see (2.4) and (2.6). We use the following elementary (and rather crude) estimates,

$$\begin{array}{ll} \left|I_{\frac{1}{2}}(x) - \sqrt{\frac{2x}{\pi}}\right| \leq \frac{1}{5}\sqrt{\frac{2x^5}{\pi}} & \text{for } 0 < x < 1, \\ 0 < I_{\frac{1}{2}}(x) \leq \frac{e^x}{\sqrt{2\pi x}} & \text{for } x > 0, \\ |K_\psi(m,n,4c)| \leq c/2 & \text{for all } c \in \mathbb{N}, \end{array}$$

and set $\delta_c = 1 + \delta_{\text{odd}}(Nc)$ which has the obvious bounds $1 \leq \delta_c \leq 2$.

The convergence and bounds of the coefficients $A_{N,\psi}$ rely on the convergence of the modified *Selberg–Kloosterman zeta function*

$$Z^*_\psi(m,n;s) := \sum_{c=1}^{\infty}(1-i)(1+\delta_{\text{odd}}(Nc))\frac{K_\psi(m,n,4Nc)}{(4Nc)^{2s}}. \tag{4.1}$$

The zeta function only converges conditionally at 3/4. The bounds we obtain are crude and very large, but they do not grow with n.

If we set

$$C_{[g]}(n) := \frac{4N}{\pi\sqrt{2}}\left(\frac{n}{3}\right)^{1/4} A_{N,\psi} = \sum_{c=1}^{\infty}(1-i)(1+\delta_{\text{odd}}(Nc))\frac{K_\psi(-3,n,4Nc)}{4Nc} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{3n}}{Nc}\right),$$

then using the triangle inequality we find

$$\begin{aligned}&|D| - |R| - \left|\sqrt{8}(3n)^{1/4}Z^*_\psi(m,n;3/4)\right| \\ &\quad \leq |C(n)| \leq |D| + |R| + \left|\sqrt{8}(3n)^{1/4}Z^*_\psi(m,n;3/4)\right|.\end{aligned}$$

Here, R is the absolutely convergent sum

$$\sum_{c=2}^{\infty}(1-i)(\delta_c)\frac{K_\psi(-3,n,4Nc)}{4Nc}\left(I_{\frac{1}{2}}\left(\frac{\pi\sqrt{3n}}{Nc}\right) - \sqrt{\frac{2\sqrt{3n}}{Nc}}\right),$$

and D is the dominant term coming from the $c = 1$ term in the expression for $A_{N,\psi}$. We may estimate $|D|$ by

$$
\begin{aligned}
|D| &= \left|(1-i)(\delta_1)\frac{K_\psi(-3,n,4N)}{4N}\left(I_{\frac{1}{2}}\left(\frac{\pi\sqrt{3n}}{N}\right)-\sqrt{\frac{2\sqrt{3n}}{N}}\right)\right| \\
&\leq \sqrt{2}\frac{\delta_1}{2}\frac{\sqrt{N}}{\pi\sqrt{2}(3n)^{\frac{1}{4}}}e^{\frac{\pi\sqrt{3n}}{N}} \\
&\leq \frac{2\sqrt{N}}{\pi(3n)^{\frac{1}{4}}}e^{\frac{\pi\sqrt{3n}}{N}}.
\end{aligned}
$$

Here, we have used the second estimate for the Bessel function. We will only be interested in a *lower* bound for $|D|$ when $[g] = [1A]$. In this case, we will just use the exact expression

$$D_{[1A]} = (-1)^n\left(I_{\frac{1}{2}}\left(\pi\sqrt{3n}\right)-\sqrt{2}(3n)^{1/4}\right).$$

If we set $L := \frac{\pi}{N}\sqrt{3n}$, the sum for $2 \leq c \leq L$ in R can be estimated as follows:

$$
\begin{aligned}
&\left|(1-i)\sum_{2\leq c\leq L}\delta_c\frac{K_\psi(-3,n,4Nc)}{4Nc}\left(I_{\frac{1}{2}}\left(\frac{\pi\sqrt{3n}}{Nc}\right)-\sqrt{\frac{2\sqrt{3n}}{Nc}}\right)\right| \\
&\leq \sqrt{2}\sum_{2\leq c\leq L}\frac{\delta_c}{2}\frac{\sqrt{Nc}}{\pi\sqrt{2}(3n)^{\frac{1}{4}}}e^{\frac{\pi\sqrt{3n}}{Nc}} \leq \frac{\sqrt{N}}{\pi(3n)^{\frac{1}{4}}}L^{\frac{3}{2}}e^{\frac{\pi\sqrt{3n}}{2N}} \\
&= \frac{\sqrt{3\pi n}}{N}e^{\frac{\pi\sqrt{3n}}{2N}}.
\end{aligned}
$$

For the terms of R with $c \geq L$, we can use the first estimate on the Bessel function.

$$
\begin{aligned}
&\left|(1-i)\sum_{c>L}^{\infty}\delta_c\frac{K_\psi(-3,n,4Nc)}{4Nc}\left(I_{\frac{1}{2}}\left(\frac{\pi\sqrt{3n}}{Nc}\right)-\sqrt{\frac{2\sqrt{3n}}{Nc}}\right)\right| \\
&\leq \sqrt{2}\left|\sum_{c>L}^{\infty}\frac{\delta}{2}\frac{\sqrt{2}\pi^2(3n)^{\frac{5}{4}}}{5(Nc)^{\frac{5}{4}}}\right| \\
&\leq \frac{2\pi^2(3n)^{\frac{5}{4}}}{5N^{\frac{5}{2}}}\zeta\left(\frac{5}{2}\right),
\end{aligned}
$$

where $\zeta(s)$ denotes the Riemann zeta function. We now need only estimate $Z_\psi^*(m,n;\frac{3}{4})$. It turns out that we can use Gannon's estimates almost directly once we write the modified zeta function in a form sufficiently similar to the expressions he uses in his estimates. However, we will need to slightly modify some of Harvey and Rayhaun's notation for v and h. Let $\hat{h} = \frac{h}{(h,4)}$ and $\hat{v} = \frac{4v}{(h,4)} \pmod{\hat{h}}$ so that $\hat{h} \mid (N,24)$ and

$$\psi(4Nc,d) = \exp\left(-2\pi i\hat{v}\frac{cd}{\hat{h}}\right).$$

We note that in every case given we have that $\hat{v} \equiv \pm 1 \pmod{\hat{h}}$.

With this notation, we have that

Proposition 4.1 *Let $n \geq 40$ with $D = mn$ a negative discriminant. The Selberg–Kloosterman zeta function defined in (4) converges at $s = 3/4$, with the following bounds.*

If $N \neq 2$, then

$$|Z_\psi^*(m,n;3/4)| \leq \frac{1}{4}\left(\prod_{p|4N\hat{h}}\left(1+\frac{1}{p}\right)\right)(1+2.13|D|^{1/8}\log|D|) \times \Big((6.124N^{35/6}\hat{h}^{47/6} - 3.09N^{23/4}\hat{h}^{31/4} + 64.32N^{29/6}\hat{h}^7 - 23N^{19/4}\hat{h}^7)|D| + \Big(.146N^{47/6}\hat{h}^{65/6} - .114N^{31/4}\hat{h}^{43/4} + 2.51N^{35/6}\hat{h}^{10} - .74N^{23/4}\hat{h}^{10}\Big)|D|^{3/2}\Big).$$

If $N = 2$, then

$$|Z_\psi(m,n;3/4)| \leq \frac{1}{4}\left(\prod_{p|4N\hat{h}}\left(1+\frac{1}{p}\right)\right)(1+2.13|D|^{1/8}\log|D|) \times \left(3872|D| + 213|D|^{3/2}\right).$$

Proof Combining Eqs. (2.3) and (2.2), we can write the Kloosterman sum $K_\psi(m,n,4Nc)$ as

$$K_\psi(m,n,4Nc) = \sum_{d \pmod{4Nc}}^{*} \left(\frac{4Nc}{d}\right)\varepsilon_d \exp\left[2\pi i\left(\frac{m\overline{d} + \left(n - 4\hat{v}c^2\cdot\frac{N}{\hat{h}}\right)d}{4Nc}\right)\right]. \tag{4.2}$$

Using a result by Kohnen [28, Proposition 5], we can write our Kloosterman sum as a sum over a more sparse set. Kohnen shows that

$$\frac{1}{\sqrt{4Nc}}(1-i)\left(1+\delta_{\text{odd}}(Nc)\right)K(m,n,4Nc) = \sum_{\substack{\beta \pmod{2Nc} \\ \beta^2\equiv mn \pmod{4Nc}}} \chi_m\left(\left[Nc,\beta,\frac{\beta^2-mn}{4Nc}\right]\right)\exp\left[2\pi i\left(\frac{\beta}{2Nc}\right)\right]. \tag{4.3}$$

Here, $[\alpha,\beta,\gamma]$ is a positive definite integral binary quadratic form, in this case with discriminant mn, and χ_Δ is the genus character defined as follows on integral binary quadratic forms with discriminant divisible by Δ by

$$\chi_\Delta(Q) = \chi_\Delta([\alpha,\beta,\gamma]) := \begin{cases} \left(\frac{\Delta}{R}\right) & \text{if } \gcd(\Delta,\alpha,\beta,\gamma)=1 \text{ where } Q \text{ represents } R \\ 0 & \text{otherwise.} \end{cases}$$

We can write Eq. (4.2) in this form if we replace n with $\tilde{n} = n - 4\hat{v}c^2\cdot\frac{N}{\hat{h}}$. Unfortunately, this makes the sum over a set of quadratic forms with discriminant $m\tilde{n}$ which depends on c. This is not ideal for approximating the zeta function. To fix this, notice that if the quadratic form $Q = [Nc,\beta,\gamma]$ has discriminant $m\tilde{n}$, then the form $Q' = [Nc,\beta,\gamma'/\hat{h}]$ with $\gamma' = \frac{\beta^2-mn}{4Nc/\hat{h}}$ is a positive definite binary quadratic form, with discriminant mn and $\gamma' \equiv mvc \pmod{\hat{h}}$. This relation defines a bijection between such forms.

Let $\mathcal{Q}_{N;\hat{h},m\hat{v}}(D)$ denote the set of quadratic forms $Q = [Nc, \beta, \gamma/\hat{h}]$ of discriminant D with $c, \beta, \gamma \in \mathbb{Z}$ and $\gamma \equiv m\hat{v}c \pmod{\hat{h}}$, and let $\mathcal{Q}_N(d)$ denote the set of quadratic forms $Q = [Nc, \beta, \gamma]$ of discriminant d. Then, we have the bijection

$$\varphi_{\hat{h},m\hat{v}}: \mathcal{Q}_{N;\hat{h},m\hat{v}}(D) \to \mathcal{Q}_{N;}\left(D - 4m\hat{v}\alpha^2 N/\hat{h}\right)$$

defined by

$$\varphi_{\hat{h},m\hat{v}}\left[N\alpha, \beta, \gamma/\hat{h}\right] = \left[N\alpha, \beta, (\gamma - m\hat{v}\alpha)/\hat{h}\right].$$

We may drop the subscript of φ as it will generally be clear from context.

The set $\mathcal{Q}_{N;\hat{h},m\hat{v}}(D)$ is acted upon by a certain matrix group which we denote by $\Gamma_0(N; h, m\hat{v})$. This group consists of matrices $\begin{pmatrix} a & b/h \\ Nc & d \end{pmatrix}$ of determinant 1 where each letter is an integer satisfying the relations

$$a \equiv \ell d \pmod{\hat{h}} \quad \text{and} \quad b \equiv \ell m\hat{v}c \pmod{\hat{h}}.$$

Here, ℓ is some number coprime to $\hat{h}$. This generalizes the groups $\Gamma_0(N; h) = \Gamma_0(N; h, 1)$ used by Gannon.

Proposition 4.2 *Assume the notation above, and let $Q_1, Q_2 \in \mathcal{Q}_{N;\hat{h},\Delta\hat{v}}(D)$ where Δ, D are discriminants, with Δ fundamental and D divisible by Δ. If Q_1 and Q_2 are related by the action of some $M \in \Gamma_0(N; h, m\hat{v})$, then*

$$\chi_\Delta(\varphi Q_1) = \chi_\Delta(\varphi Q_2)$$

Proof Since Δ is a discriminant, the definition of $\chi_\Delta(Q)$ as a Kronecker symbol allows us to reduce the coefficients of the quadratic form Q modulo Δ. It is also multiplicative. If $\Delta = \Delta_1\Delta_2$ is a factorization into discriminants, then

$$\chi_\Delta(Q) = \chi_{\Delta_1}(Q)\chi_{\Delta_2}(Q).$$

We will want a factorization of Δ into discriminants $\Delta = \Delta'\Delta_h$ where $(\Delta', \hat{h}) = 1$ and Δ_h is divisible only by primes dividing $\hat{h}$. Since Δ is a fundamental discriminant and $\hat{h}$ divides 24, this means that $|\Delta_h|$ also divides 24.

By construction, $\varphi\, Q_1 \equiv Q_1 \pmod{\Delta'}$. As a determinant 1 matrix will not alter the integers represented by a quadratic form, we have that

$$\chi_{\Delta'}(\varphi\, Q_1) = \chi_{\Delta'}(Q_1) = \chi_{\Delta'}(Q_2) = \chi_{\Delta'}(\varphi\, Q_2)$$

Suppose $Q_1 = [N\alpha, \beta, \gamma/\hat{h}]$ and $Q_2 = Q_1|M$ where M is the matrix $\begin{pmatrix} a & b/\hat{h} \\ Nc & d \end{pmatrix}$ with $a \equiv \ell d \pmod{\hat{h}}$. If we set $\gamma' = (\gamma - m\hat{v}\alpha)/\hat{h}$, then a short calculation shows that

$$\varphi Q_2 \equiv \left[N(\alpha a^2 + \beta ac), \beta ad, d^2\gamma'\right] \pmod{\Delta}.$$

Since M has determinant 1, we have that a and d are coprime to Δ. Since $\Delta_h \mid 24$, we have that $a^2 \equiv d^2 \equiv 1 \pmod{\Delta_h}$. Moreover, by considering the discriminant, we see that β is even if Δ_h is, and $4 \mid \beta$ if $8 \mid \Delta_h$. In either case, we find that Δ_h divides βN. Therefore, we may further reduce to

$$\varphi Q_2 \equiv [N\alpha, \beta\ell, \gamma'] \pmod{\Delta_h}.$$

The ℓ does not change the possible numbers represented, and so we have that

$$\chi_{\Delta_h}(\varphi\, Q_1) = \chi_{\Delta_h}(\varphi\, Q_2),$$

concluding the proof of Proposition 4.2. □

Once again following Gannon, if we combine Eqs. (4.1), (4.3) and Proposition 4.2, we can write the modified Selberg–Kloosterman zeta function at $s = 3/4$ as

$$Z^*_\psi(m, n; 3/4) = \lim_{X\to\infty} \sum_{[Q]\in \mathcal{Q}_{N;\hat{h},\hat{v}}(mn)/\Gamma_0(N;\hat{h},m\hat{v})} \frac{\chi_m(\varphi_{\hat{h},m\hat{v}}Q)}{\omega_Q} \sum_{\substack{r,Ns\in Z,\\ (r,Ns/h)=1\\ 0<c(Q,r,Ns)<X}} \frac{1}{4Nc(Q, r, Ns)} \exp\left(2\pi i \frac{\beta}{2Nc(Q, r, Ns)}\right).$$

Here, $Q = [N\alpha, \beta, \gamma/\hat{h}]$, ω_Q is the order of the stabilizer of Q in $\Gamma_0(N; \hat{h}, m\hat{v})$, and $c(Q, r, Ns) = \frac{Q(r,Ns)}{4N}$.

This equation is analogous to Equation (4.26) of [23], but differs in four main points: First, we have normalized the zeta function slightly differently. Second, the bijection $\varphi_{\hat{h},m\hat{v}}$ and Proposition 4.2 give a more general version of Gannon's Lemma 5(b) allowing us to sum over $\mathcal{Q}_{N;\hat{h},\hat{v}}(mn)/\Gamma_0(N; \hat{h}, m\hat{v})$ rather than $\mathcal{Q}_{N;\hat{h}}(mn)/\Gamma_0(N; \hat{h})$. Third, Gannon's case was restricted to discriminants where the stabilizer could only be $\{\pm I\}$, and so he replaces the ω_Q term with a 2 in his equation. We will use this as a lower bound for ω_Q. Fourth, his sum contains a power of -1 while ours contains a genus character. In either case, the sign is constant for a given representative quadratic form Q.

Gannon estimates the inner sums in absolute value and the outer sum by bounding the number of classes of quadratic forms. His bounds for the size of $\mathcal{Q}_{N;\hat{h}}(mn)/\Gamma_0(N; \hat{h})$ are crude enough to also hold for the number of classes of $\mathcal{Q}_{N;\hat{h},\hat{v}}(mn)/\Gamma_0(N; \hat{h}, m\hat{v})$. Proposition 4.1 follows from using Gannon's bounds modified only to account for our differences in normalization. □

Combining these estimates as described above, we find that each multiplicity of the irreducible components of W_n must alway be positive for $n \geq 375$. Explicit calculations up to $n = 375$ show that these multiplicities are always positive. The worst cases for the estimates with $n \leq 375$ arise from the trivial character or from estimating for Selberg–Kloosterman zeta function for the 24*CD* conjugacy class. These calculations were performed using Sage mathematical software [39].

5 Replicability

One important property of the Hauptmoduln occurring in Monstrous Moonshine is that they are *replicable*.

Definition 5.1 Let $f(\tau) = q^{-1} + \sum_{n=0}^{\infty} H_n q^n$ be a (formal) power series with integer coefficients and consider the function

$$F(\tau_1,\tau_2)=\log\left(f(\tau_1)-f(\tau_2)\right)=\log\left(q_1^{-1}-q_2^{-1}\right)-\sum_{m,n=1}^{\infty}H_{m,n}q_1^m q_2^n,$$

where $\tau_1,\tau_2\in\mathfrak{H}$ are two independent variables and $q_j=e^{2\pi i\tau_j}$, $j=1,2$. We call f *replicable*, if we have that $H_{a,b}=H_{c,d}$ whenever $ab=cd$ and $\gcd(a,b)=\gcd(c,d)$.

This property of the Hauptmoduln involved in Monstrous Moonshine in a sense reflects the algebra structure of the Monstrous Moonshine module, see [14,30].

An important, but not immediately obvious fact is that any replicable function is determined by its first 23 Fourier coefficients, [21,30].

Theorem 5.2 *Let $f(\tau)=q^{-1}+\sum_{n=1}^{\infty}a_nq^n$ be a replicable function. Then one can compute the coefficient a_n for any $n\in\mathbb{N}$ constructively out of the coefficients*

$$\{a_1,a_2,a_3,a_4,a_5,a_7,a_8,a_9,a_{11},a_{17},a_{19},a_{23}\}.$$

A Maple procedure to perform this computation is printed at the end of [21].

In [18], there is also an analogous notion of replicability in the mock modular sense, which requires that the Fourier coefficients satisfy a certain type of recurrence. This is a special phenomenon occurring for mock theta functions, i.e., mock modular forms whose shadow is a unary theta function, satisfying certain growth conditions at cusps, see [26,29].

It is now natural to ask about replicability properties of the McKay–Thompson series in the case of Thompson Moonshine. Let g be any element of the Thompson group and $\mathcal{T}_{[g]}(\tau)=\mathcal{F}_{[g]}(\tau)$ be the corresponding McKay–Thompson series as in (2.6). As we have seen, this is a weekly holomorphic modular form of weight $\frac{1}{2}$ living in the Kohnen plus space. To relate these to the Hauptmoduln and other replicable functions discussed in [21], we split into an even and an odd part

$$\mathcal{T}_{[g]}^{(0)}(\tau)=\sum_{m=0}^{\infty}\mathrm{strace}_{W_{4m}}q^{4m}\quad\text{and}\quad\mathcal{T}_{[g]}^{(0)}(\tau)=\sum_{m=0}^{\infty}\mathrm{strace}_{W_{4m-3}}q^{4m-3}$$

in the notation of Conjecture 1.1. Letting

$$\vartheta^{(0)}(\tau)=\vartheta(\tau)=\frac{\eta(2\tau)^5}{\eta(\tau)^2\eta(4\tau)^2}=\sum_{n\in\mathbb{Z}}q^{n^2}$$

and

$$\vartheta^{(1)}(\tau)=2\frac{\eta(4\tau)^2}{\eta(2\tau)}=\sum_{n\in\mathbb{Z}}q^{\left(n+\frac{1}{2}\right)^2}$$

we define the weight 0 modular functions

$$\mathbf{t}_{[g]}^{(j)}(\tau)=\frac{\mathcal{T}_{[g]}^{(j)}\left(\frac{\tau}{4}\right)}{\vartheta^{(j)}(\tau)},\quad (j=0,1)$$

and we set $\mathbf{t}_{[g]}(\tau)=\mathbf{t}_{[g]}^{(0)}(\tau)+\mathbf{t}_{[g]}^{(1)}(\tau)$. Note that these functions do not have poles in $\mathfrak{H}$.

As it turns out through direct inspection, these weight 0 functions are often replicable functions or univariate rational functions therein. We used the list of replicable functions given in [21] as a reference and found the identities given in Tables 6, which are all identities of the given form using the aforementioned table of replicable functions at the end of [21] and allowing the degree of the denominator of the rational function to be as large as 40.

Author details
[1]Department of Mathematics, Princeton University, Fine Hall, Washington Rd., Princeton, NJ 08544, USA, [2]Department of Mathematics and Computer Science, Emory University, 400 Dowman Drive, Atlanta, GA 30322, USA.

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Appendix A: Tables

In this appendix, we include some tables with data relevant to our proofs. Tables 1, 2, 3, 4 contain the character table of the Thompson group *Th* over the complex numbers.

$$\begin{array}{lll} A := -1+2i\sqrt{3}, & B := -2+4i\sqrt{3}, & C := \frac{-1+i\sqrt{15}}{2}, \\ D := -i\sqrt{3}, & E := -i\sqrt{6}, & F := \frac{-1+3i\sqrt{3}}{2}, \\ G := \frac{-1+i\sqrt{31}}{2}, & H := -1+i\sqrt{3}, & I := \frac{-1+i\sqrt{39}}{2}, \end{array}$$

and overlining one of these characters denotes complex conjugation. We used Gap4 [38] to find the character table.

Table 5 contains the multiplier systems associated to each conjugacy class, the necessary theta correction, as well as the levels $N_{[g]}$, and in Table 6, we give representation of the weight 0 functions introduced in Sect. 5 in terms of replicable functions.

Table 1 Character table of *Th*, Part I

	1A	2A	3A	3B	3C	4A	4B	5A	6A	6B	6C	7A	8A	8B	9A	9B	9C	10A	12A	12B	12C	12D	13A	14A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	248	−8	14	5	−4	8	0	−2	4	−2	1	3	0	0	5	−4	2	2	2	2	−1	0	1	−1
χ_3	4123	27	64	−8	1	27	−5	−2	9	0	0	7	3	−1	−8	1	4	2	0	0	0	1	2	−1
χ_4	27,000	120	−27	27	0	8	0	0	0	−3	3	1	0	0	0	0	0	0	A	$\overline{A}$	−1	0	−1	1
χ_5	27,000	120	−27	27	0	8	0	0	0	−3	3	1	0	0	0	0	0	0	$\overline{A}$	A	−1	0	−1	1
χ_6	30,628	−92	91	10	10	36	4	3	10	−5	−2	3	−4	0	10	10	1	3	3	3	0	−2	0	−1
χ_7	30,875	155	104	14	5	27	−5	0	5	8	2	5	3	−1	14	5	2	0	0	0	0	1	0	1
χ_8	61,256	72	182	20	20	56	0	6	12	6	0	6	0	0	−7	−7	2	2	2	2	2	0	0	2
χ_9	85,995	−21	0	−27	27	−21	11	−5	3	0	−3	0	3	−1	0	0	0	−1	0	0	−3	−1	0	0
χ_{10}	85,995	−21	0	−27	27	−21	11	−5	3	0	−3	0	3	−1	0	0	0	−1	0	0	−3	−1	0	0
χ_{11}	147,250	50	181	−8	−35	34	10	0	5	5	−4	5	2	−2	19	−8	1	0	1	1	−2	1	−1	1
χ_{12}	767,637	405	0	0	0	−27	−3	12	0	0	0	3	−3	−3	0	0	0	0	0	0	0	0	0	−1
χ_{13}	767,637	405	0	0	0	−27	−3	12	0	0	0	3	−3	−3	0	0	0	0	0	0	0	0	0	−1
χ_{14}	779,247	−273	−189	−54	0	63	−9	−3	0	3	6	0	−1	3	0	0	0	−3	3	3	0	0	1	0
χ_{15}	779,247	−273	−189	−54	0	63	−9	−3	0	3	6	0	−1	3	0	0	0	−3	3	3	0	0	1	0
χ_{16}	957,125	−315	650	−52	−25	133	5	0	15	−6	0	8	−3	1	−25	2	2	0	−2	−2	−2	−1	0	0
χ_{17}	1,707,264	−768	0	−54	54	0	0	14	−6	0	6	6	0	0	0	0	0	2	0	0	0	0	0	2
χ_{18}	1,707,264	−768	0	−54	54	0	0	14	−6	0	6	6	0	0	0	0	0	2	0	0	0	0	0	2
χ_{19}	2,450,240	832	260	71	44	64	0	−10	4	4	−5	−5	0	0	17	−10	−1	2	4	4	1	0	0	−1
χ_{20}	2,572,752	−1072	624	111	84	48	0	2	−4	−16	−1	7	0	0	30	3	3	−2	0	0	3	0	0	−1
χ_{21}	3,376,737	609	819	9	9	161	1	−13	9	3	−3	0	1	1	9	9	0	−1	−1	−1	−1	1	0	0
χ_{22}	4,096,000	0	64	−8	−80	0	0	0	0	0	0	−8	0	0	−8	1	4	0	0	0	0	0	−1	0
χ_{23}	4,096,000	0	64	−8	−80	0	0	0	0	0	0	−8	0	0	−8	1	4	0	0	0	0	0	−1	0
χ_{24}	4,123,000	120	118	19	−80	8	0	0	0	6	3	−7	0	0	19	1	4	0	2	2	−1	0	−2	1

Table 2 Character table of *Th*, Part II

	1A	2A	3A	3B	3C	4A	4B	5A	6A	6B	6C	7A	8A	8B	9A	9B	9C	10A	12A	12B	12C	12D	13A	14A
χ_{25}	4,881,384	1512	729	0	0	72	24	9	0	9	0	4	8	0	0	0	0	−3	−3	−3	0	0	1	0
χ_{26}	4,936,750	−210	637	−38	−65	126	−10	0	15	−3	6	0	−2	2	16	−11	−2	0	−3	−3	0	−1	0	0
χ_{27}	6,669,000	−1080	−351	108	0	56	0	0	0	9	0	2	0	0	0	0	0	0	A	$\overline{A}$	2	0	0	−2
χ_{28}	6,669,000	−1080	−351	108	0	56	0	0	0	9	0	2	0	0	0	0	0	0	$\overline{A}$	A	2	0	0	−2
χ_{29}	6,696,000	−960	−378	135	0	64	0	0	0	6	3	3	0	0	0	0	0	0	B	$\overline{B}$	1	0	−1	−1
χ_{30}	6,696,000	−960	−378	135	0	64	0	0	0	6	3	3	0	0	0	0	0	0	$\overline{B}$	B	1	0	−1	−1
χ_{31}	10,822,875	−805	924	141	−75	91	−5	0	5	−4	5	0	3	−1	−21	6	−3	0	4	4	1	1	−2	0
χ_{32}	11,577,384	552	351	135	0	−120	24	9	0	15	3	7	−8	0	0	0	0	−3	3	3	−3	0	0	−1
χ_{33}	16,539,120	2544	0	297	−54	48	16	−5	−6	0	−3	3	0	0	0	0	0	−1	0	0	3	−2	0	3
χ_{34}	18,154,500	1540	−273	213	−30	−28	20	0	10	−17	1	0	−4	0	−3	−3	−3	0	−1	−1	−1	2	0	0
χ_{35}	21,326,760	168	0	−135	−108	−168	0	10	12	0	−3	0	0	0	0	0	0	−2	0	0	3	0	0	0
χ_{36}	21,326,760	168	0	−135	−108	−168	0	10	12	0	−3	0	0	0	0	0	0	−2	0	0	3	0	0	0
χ_{37}	28,861,000	840	1078	−110	160	56	0	0	0	6	−6	0	0	0	−29	−2	−2	0	2	2	2	0	−1	0
χ_{38}	30,507,008	0	896	−184	32	0	0	8	0	0	0	0	0	0	32	5	−4	0	0	0	0	0	−1	0
χ_{39}	40,199,250	3410	−78	3	165	−62	10	0	5	2	−1	−7	−6	2	3	3	3	0	−2	−2	1	1	0	1
χ_{40}	44,330,496	3584	168	6	−156	0	0	−4	−4	8	2	0	0	0	6	6	−3	4	0	0	0	0	2	0
χ_{41}	51,684,750	2190	0	108	135	−162	−10	0	15	0	12	−9	6	−2	0	0	0	0	0	0	0	−1	0	−1
χ_{42}	72,925,515	−2997	0	0	0	27	51	15	0	0	0	−9	3	3	0	0	0	3	0	0	0	0	0	−1
χ_{43}	76,271,625	−2295	729	0	0	153	−15	0	0	9	0	−11	−7	−3	0	0	0	0	−3	−3	0	0	1	1
χ_{44}	77,376,000	2560	1560	−60	−60	0	0	0	−20	−8	4	2	0	0	−6	0	0	−3	3	3	0	0	2	1
χ_{45}	81,153,009	−783	−729	0	0	225	9	9	0	−9	0	−7	1	−3	0	0	0	−3	3	3	0	0	2	1
χ_{46}	91,171,899	315	0	243	0	−21	−45	24	0	0	−9	0	3	3	0	0	0	0	0	0	−3	0	0	0
χ_{47}	111,321,000	3240	−1728	−216	0	216	0	0	0	0	0	7	0	0	0	0	0	0	0	0	0	0	−2	−1
χ_{48}	190,373,976	−3240	0	0	0	−216	0	−24	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	1

Table 3 Character table of *Th*, Part III

	15A	15B	18A	18B	19A	20A	21A	24A	24B	24C	24D	27A	27B	27C	28A	30A	30B	31A	31B	36A	36B	36C	39A	39B
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	−2	1	0	0	0	0	0	0	2	−1	−1	1	−1	−1	0	0	−1	−1	−1	1	1
χ_3	1	1	0	0	0	0	1	0	0	−1	−1	−2	1	1	−1	−1	−1	0	0	0	0	0	−1	−1
χ_4	0	0	0	0	1	0	1	D	$-D$	0	0	0	0	0	1	0	0	−1	−1	2	H	$\overline{H}$	−1	−1
χ_5	0	0	0	0	1	0	1	$-D$	D	0	0	0	0	0	1	0	0	−1	−1	2	$\overline{H}$	H	−1	−1
χ_6	0	0	−2	1	0	−1	0	−1	−1	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
χ_7	0	0	2	2	0	0	−1	0	0	−1	−1	2	−1	−1	−1	0	0	−1	−1	0	0	0	0	0
χ_8	0	0	−3	0	0	0	0	0	0	0	0	−1	−1	−1	0	2	2	0	0	−1	−1	−1	0	0
χ_9	C	$\overline{C}$	0	0	1	1	0	0	0	−1	−1	0	0	0	0	$-C$	$-\overline{C}$	1	1	0	0	0	0	0
χ_{10}	$\overline{C}$	C	0	0	1	1	0	0	0	−1	−1	0	0	0	0	$-\overline{C}$	$-C$	1	1	0	0	0	0	0
χ_{11}	0	0	−1	−1	0	0	−1	−1	−1	1	1	1	1	1	−1	0	0	0	0	1	1	1	−1	−1
χ_{12}	0	0	0	0	−1	2	0	0	0	0	0	0	0	0	1	0	0	G	$\overline{G}$	0	0	0	0	0
χ_{13}	0	0	0	0	−1	2	0	0	0	0	0	0	0	0	1	0	0	$\overline{G}$	G	0	0	0	0	0
χ_{14}	0	0	0	0	0	1	0	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	I	$\overline{I}$
χ_{15}	0	0	0	0	0	1	0	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	$\overline{I}$	I
χ_{16}	0	0	3	0	0	0	−1	0	0	1	1	−1	−1	−1	0	0	0	0	0	1	1	1	0	0
χ_{17}	−1	−1	0	0	0	0	0	0	0	E	$-E$	0	0	0	0	−1	−1	1	1	0	0	0	0	0
χ_{18}	−1	−1	0	0	0	0	0	0	0	$-E$	E	0	0	0	0	−1	−1	1	1	0	0	0	0	0
χ_{19}	−1	−1	1	1	0	0	1	0	0	0	0	−1	−1	−1	1	−1	−1	0	0	1	1	1	0	0
χ_{20}	−1	−1	2	−1	0	0	1	0	0	0	0	0	0	0	−1	1	1	0	0	0	0	0	0	0
χ_{21}	−1	−1	−3	0	0	1	0	1	1	1	1	0	0	0	0	−1	−1	0	0	−1	−1	−1	0	0
χ_{22}	0	0	0	0	−1	0	1	0	0	0	0	1	F	$\overline{F}$	0	0	0	1	1	0	0	0	−1	−1
χ_{23}	0	0	0	0	−1	0	1	0	0	0	0	1	$\overline{F}$	F	0	0	0	1	1	0	0	0	−1	−1
χ_{24}	0	0	3	0	0	0	−1	0	0	0	0	−2	1	1	1	0	0	0	0	−1	−1	−1	1	1

Table 4 Character table of *Th*, Part IV

	15A	15B	18A	18B	19A	20A	21A	24A	24B	24C	24D	27A	27B	27C	28A	30A	30B	31A	31B	36A	36B	36C	39A	39B
χ_{24}	0	0	3	0	0	0	−1	0	0	0	0	−2	1	1	1	0	0	0	0	−1	−1	−1	1	1
χ_{25}	0	0	0	0	−1	−1	1	−1	−1	0	0	0	0	0	2	0	0	0	0	0	0	0	1	1
χ_{26}	0	0	0	0	−1	0	0	1	1	−1	−1	1	1	1	0	0	0	0	0	0	0	0	0	0
χ_{27}	0	0	0	0	0	0	−1	$-D$	D	0	0	0	0	0	0	0	0	1	1	2	H	$\overline{H}$	0	0
χ_{28}	0	0	0	0	0	0	−1	D	$-D$	0	0	0	0	0	0	0	0	1	1	2	$\overline{H}$	H	0	0
χ_{29}	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−2	$-H$	$-\overline{H}$	−1	−1
χ_{30}	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−2	$-\overline{H}$	$-H$	−1	−1
χ_{31}	0	0	−1	−1	0	0	0	0	0	−1	−1	0	0	0	0	0	0	0	0	1	1	1	1	1
χ_{32}	0	0	0	0	0	−1	1	1	1	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0
χ_{33}	1	1	0	0	0	1	0	0	0	0	0	0	0	0	−1	−1	−1	0	0	0	0	0	0	0
χ_{34}	0	0	1	1	0	0	0	−1	−1	0	0	0	0	0	0	0	0	1	1	−1	−1	−1	0	0
χ_{35}	C	$\overline{C}$	0	0	1	0	0	0	0	0	0	0	0	0	0	C	$\overline{C}$	0	0	0	0	0	0	0
χ_{36}	$\overline{C}$	C	0	0	1	0	0	0	0	0	0	0	0	0	0	$\overline{C}$	C	0	0	0	0	0	0	0
χ_{37}	0	0	3	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	−1	−1	−1	−1	−1
χ_{38}	2	2	0	0	0	0	0	0	0	0	0	−1	−1	−1	0	0	0	1	1	0	0	0	−1	−1
χ_{39}	0	0	−1	−1	0	0	−1	0	0	−1	−1	0	0	0	1	0	0	0	0	1	1	1	0	0
χ_{40}	−1	−1	2	−1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	−1	−1
χ_{41}	0	0	0	0	0	0	0	0	0	1	1	0	0	0	−1	0	0	0	0	0	0	0	0	0
χ_{42}	0	0	0	0	0	1	0	0	0	0	0	0	0	0	−1	0	0	−1	−1	0	0	0	0	0
χ_{43}	0	0	0	0	1	0	1	−1	−1	0	0	0	0	0	−1	0	0	0	0	0	0	0	1	1
χ_{44}	0	0	−2	1	1	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{45}	0	0	0	0	0	−1	−1	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	−1	−1
χ_{46}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{47}	0	0	0	0	0	0	1	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	1	1
χ_{48}	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0

Table 5 Multipliers and theta corrections and levels

$[g]$	**1A**	**2A**	**3A**	**3B**	**3C**	**4A**	**4B**	**5A**	**6A**	**6B**
v, h	0, 1	0, 1	1, 3	0, 1	2, 3	0, 1	7, 8	0, 1	5, 6	2, 3
$\kappa_{m,g}$	240_1	0	$-6_1 + 18_9$	6_1	0	8_4	0	0	0	0
$N_{[g]}$	4	8	36	12	36	16	32	20	72	72
$[g]$	**6C**	**7A**	**8A**	**8B**	**9A**	**9B**	**9C**	**10A**	**12AB**	**12C**
v, h	0, 1	0, 1	7, 8	13, 16	0, 1	0, 1	1, 3	0, 1	7, 12	0, 1
$\kappa_{m,g}$	0	2_1	0	0	6_9	-3_9	0	0	$-1_4 + 3_{36}$	-1_4
$N_{[g]}$	24	28	64	128	36	36	108	40	144	48
$[g]$	**12D**	**13A**	**14A**	**15AB**	**18A**	**18B**	**19A**	**20A**	**21A**	**24AB**
v, h	19, 24	0, 1	0, 1	1, 3	0, 1	2, 3	0, 1	7, 8	1, 3	19, 24
$\kappa_{m,g}$	0	$\left(\frac{1}{3}\right)_1$	0	0	0	0	$\left(\frac{3}{5}\right)_1$	0	$\left(\frac{1}{8}\right)_1 - \left(\frac{3}{8}\right)_9$	0
$N_{[g]}$	288	52	56	180	72	216	76	160	252	576
$[g]$	**24CD**	**27A**	**27BC**	**28A**	**30AB**	**31AB**	**36A**	**36BC**	**39AB**	
v, h	37, 48	1, 3	1, 3	0, 1	2, 3	0, 1	0, 1	0, 1	1,3	
$\kappa_{m,g}$	0	$-1_9 + 3_{81}$	$\left(\frac{1}{2}\right)_9 - \left(\frac{3}{2}\right)_{81}$	1_4	0	$-\left(\frac{1}{4}\right)_1$	$2_4 - 3_{36}$	-1_4	$-\left(\frac{3}{7}\right)_1 + \left(\frac{9}{7}\right)_9$	
$N_{[g]}$	1152	324	324	112	360	124	144	144	468	

The notation $(\kappa_{m,g})_m$ indicates the addition of the theta correction $\sum_m \kappa_{m,g}\vartheta(m\tau)$ in the definition of $\mathcal{F}_{[g]}(\tau)$ in (2.6), see Table 5 in [25] and also Remark 2.3

Table 6 Relations to replicable functions

$[g]$	Replicable function	Expression
1A		
$\mathbf{t}_{[g]}^{(0)}$	$\frac{1^8}{4^8}$	$\frac{248x^4+57{,}472x^3+3{,}735{,}552x^2+79{,}691{,}776x+536{,}870{,}912}{x^4+16x^3}$
$\mathbf{t}_{[g]}^{(1)}$	$\frac{1^8}{4^8}$	$x + 8 - 86{,}016x^{-1} - 3{,}407{,}872x^{-2} - 33{,}554{,}432x^{-3}$
$\mathbf{t}_{[g]}$	$\frac{1^{24}}{2^{24}}$	$x + 272 - 2^{15}x^{-1}$
2A		
$\mathbf{t}_{[g]}^{(0)}$	$\frac{2^{24}}{1^8 \cdot 4^{16}}$	$-8 + 256x^{-1}$
$\mathbf{t}_{[g]}^{(1)}$	$\frac{2^{24}}{1^8 \cdot 4^{16}}$	$x - 8$
$\mathbf{t}_{[g]}$	$\frac{1^{24}}{2^{24}}$	$x + 16$
3A		
$\mathbf{t}_{[g]}$	$\frac{2^2 \cdot 9}{1 \cdot 18^2}$	$\frac{x^6+9x^5-126x^4+450x^3-675x^2+405x}{x^5-4x^4+9x^3-15x^2+18x-9}$
3B		
$\mathbf{t}_{[g]}^{(0)}$	$\frac{3^3 \cdot 4}{1 \cdot 12^3}$	$\frac{5x^4+34x^3-240x^2+448x-256}{x^4-2x^3}$
$\mathbf{t}_{[g]}^{(1)}$	$\frac{3^3 \cdot 4}{1 \cdot 12^3}$	$x - 1 + 24x^{-1} + 32x^{-2} - 128x^{-3}$
$\mathbf{t}_{[g]}$	$\frac{2^3 \cdot 3^9}{1^3 \cdot 6^9}$	$x + 2 + 64x^{-1}$
3C		
$\mathbf{t}_{[g]}$	$\frac{2^2 \cdot 9}{1 \cdot 18^2}$	$\frac{x^7-9x^6+9x^5+171x^4-837x^3+1701x^2-1701x+729}{x^6-4x^5+9x^4-15x^3+18x^2-9x}$
4A		
$\mathbf{t}_{[g]}^{(0)}$	0	8
$\mathbf{t}_{[g]}^{(1)}$	$\frac{1^8}{4^8}$	$x + 8$
$\mathbf{t}_{[g]}$	$\frac{1^8}{4^8}$	$x + 16$
4B		
$\mathbf{t}_{[g]}^{(0)}$	0	0
$\mathbf{t}_{[g]}^{(1)}$	$\frac{2^{12}}{4^{12}}$	x
$\mathbf{t}_{[g]}$	$\frac{2^{12}}{4^{12}}$	x
5A		
$\mathbf{t}_{[g]}$	$\frac{1^3 \cdot 5}{2 \cdot 10^3}$	$\frac{x^3+10x^2+36x+80}{x^2+9x+20}$

Table 6 continued

$[g]$	Replicable function	Expression
6A		
$\mathbf{t}_{[g]}$	$\frac{2^2\cdot 9}{1\cdot 18^2}$	$\frac{x^4-18x^2+36x-27}{x^3-3x^2+3x}$
6B		
$\mathbf{t}_{[g]}$	$\frac{2^2\cdot 9}{1\cdot 18^2}$	$\frac{x^3-6x^2+9x}{x^2-3x+3}$
6C		
$\mathbf{t}_{[g]}^{(0)}$	$\frac{4^4\cdot 6^2}{2^2\cdot 12^4}$	$\frac{x+3}{x-1}$
$\mathbf{t}_{[g]}^{(1)}$	$\frac{4^4\cdot 6^2}{2^2\cdot 12^4}$	x
$\mathbf{t}_{[g]}$	$\frac{2^3\cdot 3^9}{1^3\cdot 6^9}$	$x-2$
8A		
$\mathbf{t}_{[g]}^{(0)}$	0	0
$\mathbf{t}_{[g]}^{(1)}$	$\frac{2^4}{8^4}$	x
$\mathbf{t}_{[g]}$	$\frac{2^4}{8^4}$	x
8B		
$\mathbf{t}_{[g]}^{(0)}$	0	0
$\mathbf{t}_{[g]}^{(1)}$	$\frac{4^6}{8^6}$	x
$\mathbf{t}_{[g]}$	$\frac{4^6}{8^6}$	x
9A		
$\mathbf{t}_{[g]}$	$\frac{2^2\cdot 9}{1\cdot 18^2}$	$\frac{x^3+3x^2-15x+27}{x^2-x}$
9B		
$\mathbf{t}_{[g]}$	$\frac{2^2\cdot 9}{1\cdot 18^2}$	$\frac{x^4-6x^3+12x^2-18x+27}{x^3-x^2}$
10A		
$\mathbf{t}_{[g]}^{(0)}$	$\frac{2^4\cdot 10^4}{1^3\cdot 4\cdot 5\cdot 20^3}$	$2-4x^{-1}$
$\mathbf{t}_{[g]}^{(1)}$	$\frac{1^2\cdot 10^5}{2\cdot 4\cdot 5^2\cdot 20^3}$	$x+2$
$\mathbf{t}_{[g]}$	$\frac{2^4\cdot 5^2}{1^2\cdot 10^4}$	$x-5x^{-1}$
12AB		
$\mathbf{t}_{[g]}^{(0)}$	$\frac{1\cdot 4\cdot 18}{2\cdot 9\cdot 36}$	$\frac{2x}{x-3}$
12C		
$\mathbf{t}_{[g]}^{(0)}$	0	-1
$\mathbf{t}_{[g]}^{(1)}$	$\frac{4^4\cdot 6^2}{2^2\cdot 12^4}$	x
$\mathbf{t}_{[g]}$	$\frac{4^4\cdot 6^2}{2^2\cdot 12^4}$	$x-1$
12D		
$\mathbf{t}_{[g]}^{(0)}$	0	0
$\mathbf{t}_{[g]}^{(1)}$	$\frac{6^4}{12^4}$	x
$\mathbf{t}_{[g]}$	$\frac{6^4}{12^4}$	x
14A		
$\mathbf{t}_{[g]}^{(0)}$	$\frac{2^3\cdot 14^3}{1\cdot 4^2\cdot 7\cdot 28^2}$	$-1+4x^{-1}$
$\mathbf{t}_{[g]}^{(1)}$	$\frac{2^3\cdot 14^3}{1\cdot 4^2\cdot 7\cdot 28^2}$	$x-1$
$\mathbf{t}_{[g]}$	$\frac{1^3\cdot 7^3}{2^3\cdot 14^3}$	$x+2$
18A		
$\mathbf{t}_{[g]}^{(0)}$	$\frac{4\cdot 9}{1\cdot 36}$	$\frac{x-1}{x+1}$
$\mathbf{t}_{[g]}^{(1)}$	$\frac{2\cdot 12^4\cdot 18}{4^2\cdot 6^2\cdot 36^2}$	x
$\mathbf{t}_{[g]}$	$\frac{2^2\cdot 9}{1\cdot 18^2}$	$x-3x^{-1}$

Table 6 continued

[g]	Replicable function	Expression
20A		
$\mathbf{t}_{[g]}^{(0)}$	0	0
$\mathbf{t}_{[g]}^{(1)}$	$\frac{2^2\cdot 10^2}{4^2\cdot 20^2}$	x
$\mathbf{t}_{[g]}$	$\frac{2^2\cdot 10^2}{4^2\cdot 20^2}$	x
24AB		
$\mathbf{t}_{[g]}$	0	0
24CD		
$\mathbf{t}_{[g]}^{(0)}$	0	0
$\mathbf{t}_{[g]}^{(1)}$	$\frac{12^2}{24^2}$	x
$\mathbf{t}_{[g]}$	$\frac{12^2}{24^2}$	x
28A		
$\mathbf{t}_{[g]}^{(0)}$	0	1
$\mathbf{t}_{[g]}^{(1)}$	$\frac{1\cdot 7}{4\cdot 28}$	$x+1$
$\mathbf{t}_{[g]}$	$\frac{1\cdot 7}{4\cdot 28}$	$x+2$
36A		
$\mathbf{t}_{[g]}^{(0)}$	$\frac{2^5\cdot 3\cdot 12\cdot 18}{1^2\cdot 4^2\cdot 6^2\cdot 9\cdot 36}$	$-1+6x^{-1}$
$\mathbf{t}_{[g]}^{(1)}$	$\frac{2\cdot 12^4\cdot 18}{4^2\cdot 6^2\cdot 36^2}$	x
$\mathbf{t}_{[g]}$		$\mathbf{t}_{[g]}^{(0)}+t_{[g]}^{(1)}$
36BC		
$\mathbf{t}_{[g]}^{(0)}$	0	-1
$\mathbf{t}_{[g]}^{(1)}$	$\frac{2\cdot 12^4\cdot 18}{4^2\cdot 6^2\cdot 36^2}$	x
$\mathbf{t}_{[g]}$	$\frac{1\cdot 12\cdot 18^3}{4\cdot 6\cdot 9\cdot 36^2}$	x

In the table's third column, we use the short hand d^r for the expression $\eta(d\tau)^r$. Thus, for example the expression $\frac{1^8}{4^8}$ stands for $\eta(\tau)^8\eta(4\tau)^{-8}$. The rational function in the fourth column means that the replicable function from the third column plugged into it gives the functions indicated in the second column

Appendix B: Congruences

Here, we give the linear relations and congruences for the McKay–Thompson series of Thompson moonshine. In the relations below, the symbol $[g]$ represents the McKay–Thompson series $\mathcal{T}_{[g]}$, and the symbol $[\vartheta_{m^2}]$ represents $\vartheta(m^2\tau)$.

B.1 Linear relations

$$\begin{aligned}0 &= 3[9A]-[3A]-[3C]-[3B]\\ &= [9B]-[9A]+9[\vartheta_9]\\ &= 3[18A]-[6A]-[6B]-[6C]\\ &= [27BC]-[27A]-3/2[\vartheta_9]-9/2[\vartheta_{81}]\\ &= [36BC]-[36A]+3[\vartheta_4]+3[\vartheta_{36}]\end{aligned}$$

B.2 Congruences

p=31:

$$0\equiv[1A]-[31AB] \pmod{31}$$

p=19:

$$0\equiv[1A]-[19A] \pmod{19}$$

p=13:

$$\begin{aligned} 0 &\equiv [1A] - [13A] && \pmod{13} \\ &\equiv [3A] - [39AB] && \pmod{13} \end{aligned}$$

p=7:

$$\begin{aligned} 0 &\equiv [1A] - [7A] && \pmod{7^2} \\ &\equiv [2A] - [14A] && \pmod{7} \\ &\equiv [3A] - [21A] && \pmod{7} \\ &\equiv [4A] - [28A] && \pmod{7} \end{aligned}$$

p=5:

$$\begin{aligned} 0 &\equiv [1A] - [5A] && \pmod{5^2} \\ &\equiv [2A] - [10A] && \pmod{5} \\ &\equiv [3C] - [15AB] && \pmod{5} \\ &\equiv [4B] - [20A] && \pmod{5} \\ &\equiv [6A] - [30AB] && \pmod{5} \end{aligned}$$

p=3:

$$\begin{aligned} 0 &\equiv [1A] - [3A] + 9[\vartheta_9] && \pmod{3^3} \\ &\equiv 28[1A] - 27[3A] - [3B] && \pmod{3^7} \\ &\equiv 53[1A] - 27[3A] - 53[3B] + 27[3C] + 3^6[\vartheta_9] && \pmod{3^8} \\ &\equiv [2A] - [6A] && \pmod{3} \\ &\equiv 2[2A] - [6A] - [6B] && \pmod{3^2} \\ &\equiv 2[2A] - 3[6A] + [6C] && \pmod{3^3} \\ &\equiv 760[1A] - 864[3A] + 212[3B] + 621[3C] - 3^6[9C] \\ &\quad - 49 \cdot 3^6[\vartheta_9] - 3^9[\vartheta_{81}] && \pmod{3^{10}} \\ &\equiv [4A] - [12AB] && \pmod{3} \\ &\equiv 4[4A] - 3[12AB] - [12C] - 9[\vartheta_4] - 18[\vartheta_{36}] && \pmod{3^3} \\ &\equiv [4B] - [12D] && \pmod{3} \\ &\equiv [5A] - [15AB] && \pmod{3} \\ &\equiv 16[2A] - 15[6A] + 8[6C] - 9[18B] && \pmod{3^4} \\ &\equiv [7A] - [21A] && \pmod{3} \\ &\equiv [8A] - [24AB] && \pmod{3} \\ &\equiv [8B] - [24CD] && \pmod{3} \\ &\equiv [3C] + 2[9C] - 3[27A] - 21[\vartheta_9] && \pmod{3^3} \\ &\equiv [10A] - [30AB] && \pmod{3} \\ &\equiv [12C] - [36A] && \pmod{3} \\ &\equiv [13A] - [39AB] && \pmod{3} \end{aligned}$$

p=2:

$$
\begin{aligned}
0 &\equiv [1A] - [2A] - 2^8[\vartheta_4] && \pmod{2^{12}}\\
&\equiv [1A] + 15[2A] - 16[4A] && \pmod{2^{13}}\\
&\equiv 11[1A] + 5[2A] - 144[4A] + 2^7[4B] - 3\cdot 2^9[\vartheta_4] && \pmod{2^{16}}\\
&\equiv [3C] - [6A] && \pmod{2^{3}}\\
&\equiv [3A] - [6B] - 8[\vartheta_4] - 8[\vartheta_3 6] && \pmod{2^{5}}\\
&\equiv [3B] - [6C] + 4[\vartheta_1] && \pmod{2^{3}}\\
&\equiv [1A] - 17[2A] - 16[4A] + -96[4B] + 2^7[8A] - 2^8[\vartheta_4] && \pmod{2^{13}}\\
&\equiv 3[1A] + 13[2A] - 112[4A] + 480[4B] - 7\cdot 2^7[8A]\\
&\quad + 2^9[8B] + 2^8[\vartheta_4] && \pmod{2^{15}}
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv [5A] - [10A] + 4[\vartheta_1] && \pmod{2^{3}}\\
&\equiv [3A] - 5[6B] + 4[12AB] + 80[\vartheta_4] + 16[\vartheta_{36}] && \pmod{2^{7}}\\
&\equiv 5[3B] + [6C] + 2[12C] + 8[\vartheta_1] && \pmod{2^{4}}\\
&\equiv 3[3C] - 5[6A] + 2[12D] && \pmod{2^{5}}\\
&\equiv [7A] - [14A] + 4[\vartheta_1] && \pmod{2^{3}}\\
&\equiv [9C] - [18B] && \pmod{2^{2}}\\
&\equiv 3[5A] + 3[10A] + 2[20A] && \pmod{2^{4}}\\
&\equiv 3[3A] + [24AB] + 5[\vartheta_4] + [\vartheta_{36}] && \pmod{2^{3}}\\
&\equiv [12D] - [24CD] && \pmod{2^{3}}\\
&\equiv 5[7A] + [14A] + 2[28A] && \pmod{2^{4}}\\
&\equiv [15AB] + [30AB] && \pmod{2^{2}}\\
&\equiv [3C] + [6A] + 2[12C] + 2[24AB] + 2[36A] + 6[\vartheta_4] + 8[\vartheta_9]\\
&\quad + 6[\vartheta_3 6] && \pmod{2^{4}}
\end{aligned}
$$

Additionally, for each g of odd order we have the congruence

$$0 \equiv [g] - \alpha_{[g]}(0) \pmod{2},$$

and of course

$$1 \equiv [\theta_{m^2}] \pmod{2}.$$

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