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The Lerch zeta function IV. Hecke operators

Jeffrey C. Lagarias^{1*} and Wen-Ching Winnie Li²

*Correspondence:
lagarias@umich.edu
¹Department of Mathematics,
University of Michigan, Ann
Arbor, MI 48109-1043, USA
Full list of author information is
available at the end of the article

Abstract

This paper studies algebraic and analytic structures associated with the Lerch zeta function. It defines a family of two-variable Hecke operators $\{T_m : m \geq 1\}$ given by $T_m(f)(a, c) = \frac{1}{m} \sum_{k=0}^{m-1} f(\frac{a+k}{m}, mc)$ acting on certain spaces of real-analytic functions, including Lerch zeta functions for various parameter values. The actions of various related operators on these function spaces are determined. It is shown that, for each $s \in \mathbb{C}$, there is a two-dimensional vector space spanned by linear combinations of Lerch zeta functions characterized as a maximal space of simultaneous eigenfunctions for this family of Hecke operators. This is an analog of a result of Milnor for the Hurwitz zeta function. We also relate these functions to a linear partial differential operator in the (a, c) -variables having the Lerch zeta function as an eigenfunction.

Keywords: Functional equation, Hurwitz zeta function, Lerch zeta function

Mathematics Subject Classification: Primary 11M35

Contents

1	Background
1.1	Hecke operators and function spaces
1.2	Motivation and results
1.3	Related work
2	Main results
2.1	Two-variable Hecke operators and R-operator
2.2	Two-dimensional Lerch eigenspace
2.3	Eigenfunction characterization of Lerch eigenspace
2.4	Further extensions
3	R-operator and differential operators
3.1	Local operators: Real-variable differential operators
3.2	Nonlocal operator: R-operator
3.3	Commutation relations
4	Two-variable Hecke operators
4.1	Twisted-periodic function spaces
4.2	R-operator conjugates of Hecke operators
4.3	Commutation relations of two-variable Hecke operators
4.4	L^p -spaces: Proof of Theorem 1
5	Lerch Eigenspaces \mathcal{E}_s and their properties
5.1	Lerch zeta function: functional equations
5.2	Lerch eigenspaces \mathcal{E}_s
5.3	Operator properties of Lerch eigenspaces \mathcal{E}_s
5.4	Analytic properties of Lerch eigenspaces \mathcal{E}_s

6 Hecke Eigenfunction characterization of Lerch Eigenspaces \mathcal{E}_s
 6.1 Milnor’s theorem for Kubert functions
 6.2 Characterization of Lerch eigenspaces \mathcal{E}_s
 7 Concluding remarks
 References

1 Background

The *Lerch zeta function* is defined by the series

$$\zeta(s, a, c) = \sum_{n=0}^{\infty} \frac{e^{2\pi ina}}{(n + c)^s} \tag{1}$$

which absolutely converges for complex variables $\Re(s) > 1$, $\Re(c) > 0$ and $\Im(a) \geq 0$. In this paper, we will restrict attention to (a, c) being real variables. In that case, excluding integer values of a and c , it conditionally converges to an analytic function for $\Re(s) > 0$ and analytically continues in the s -variable to the entire plane. When either a or c is an integer (or both), it analytically continues in s to a meromorphic function whose singularity set is at most a single simple pole, located at $s = 1$, see [28, Theorem 2.2].

Two key properties of the Lerch zeta function $\zeta(s, a, c)$ for real (a, c) , restricting to $0 < a, c < 1$, are the following.

- (1) It is an eigenfunction of a linear partial differential operator

$$D_L := \frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + c \frac{\partial}{\partial c}. \tag{2}$$

It satisfies

$$(D_L \zeta)(s, a, c) = -s \zeta(s, a, c). \tag{3}$$

This property was noted in Parts II and III ([29,30]). When restricting to real variables, we regard $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial c}$ as real differential operators. (They were treated as complex differential operators in [29] and [30].)

- (2) In particular, it satisfies two four-term functional equations encoding a discrete symmetry under (s, a, c) to $(1 - s, 1 - c, a)$. These functional equations were noted by Weil [39] under the restriction $0 < a < 1$ and $0 < c < 1$ and were studied in Part I ([28]). To state the functional equations, let

$$L^{\pm}(s, a, c) := \zeta(s, a, c) \pm e^{-2\pi ia} \zeta(s, 1 - a, 1 - c)$$

and define the completed functions

$$\hat{L}^{\pm}(s, a, c) := \pi^{-\frac{s+\epsilon}{2}} \Gamma\left(\frac{s+\epsilon}{2}\right) L^{\pm}(s, a, c),$$

where $\epsilon := \frac{1}{2}(1 - (\pm 1))$ takes values 0 or 1. Then the functional equations are

$$\hat{L}^+(s, a, c) = e^{-2\pi iac} \hat{L}^+(1 - s, 1 - c, a) \tag{4}$$

and

$$\hat{L}^-(s, a, c) = i e^{-2\pi iac} \hat{L}^-(1 - s, 1 - c, a). \tag{5}$$

In this paper, we will study a family $\{T_m : m \geq 1\}$ of “Hecke operators” which also preserve the Lerch zeta function, in the sense that it is a simultaneous eigenfunction of these operators. The (*two-variable*) Hecke operators T_m are formally defined by

$$T_m(f)(a, c) := \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{a+k}{m}, mc\right). \quad (6)$$

The operators T_m stretch coordinates in the c -direction, while each individual term on the right side of (6) contracts coordinates in the a -direction, and they (formally) commute. Two-variable operators of this form were originally introduced in 1989 by Sun [37] in a completely different context, that of covering systems of arithmetic progressions of integers. In 2000, Porubsky [34, Sect. 4] noted that the Lerch zeta function is an eigenfunction of these operators on a suitable domain. For $\Re(s) > 1$, the function $\zeta(s, a, c)$ is well defined and real-analytic in the variables (a, c) in the domain

$$\mathcal{D}_{++} := \{(a, c) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}\},$$

which is the first quadrant in the (a, c) -plane. One can now check that for $\Re(s) > 1$ the Lerch zeta function $\zeta(s, a, c)$ has on this domain the simultaneous eigenfunction property

$$T_m(\zeta)(s, a, c) = m^{-s} \zeta(s, a, c) \quad (7)$$

holding for all $m \geq 1$.

The question this paper considers is that of obtaining an extension of the operators above inside suitable function spaces, in which the Lerch zeta function will be a simultaneous eigenfunction for all $s \in \mathbb{C}$. We will show there is such an extension, and study the action of all these operators on the ensuing function spaces. We then obtain a characterization of simultaneous eigenfunction solutions of these operators in these function spaces, in the spirit of Milnor [32].

1.1 Hecke operators and function spaces

The definition of the two-variable Hecke operators incorporates a dilation in one variable that seems to require it to be defined on an unbounded domain. As introduced it makes sense most naturally on a domain like $[0, 1] \times \mathbb{R}$. Obstacles to including the Lerch zeta function in such a space for all ranges of s include:

- (1) analytic continuation will be required in the s -variable to cover all $s \in \mathbb{C}$;
- (2) the functional equations do not leave the domain \mathcal{D}_{++} invariant;
- (3) the analytically continued Lerch zeta function has discontinuities at parameter values (a, c) where a is an integer or c is a nonpositive integer.

The function spaces we consider must therefore incorporate some discontinuous functions.

Our construction builds on the real-variable results obtained in Part I [28], which gave an extension of the Lerch zeta function to real variables (a, c) in $\mathbb{R} \times \mathbb{R}$. The first step was to establish properties for values of (a, c) in the closed unit square

$$\square := \{(a, c) : 0 \leq a \leq 1, 0 \leq c \leq 1\},$$

and then followed by introducing a real-analytic extension $\zeta_*(s, a, c)$ of $\zeta(s, a, c)$ to $(a, c) \in \mathbb{R} \times \mathbb{R}$ for $\Re(s) > 1$, called the *extended Lerch zeta function*, defined by

$$\zeta_*(s, a, c) = \sum_{n+c>0} e^{2\pi i na} |n+c|^{-s} \quad (8)$$

for $\Re(s) > 1$. (The resulting function is real-analytic in the variables a and c separately, away from integer values of a and c .) The extended Lerch zeta function specializes ¹ for $0 < c < 1$ to

$$\zeta_{\mathbb{R}}(s, a, c) := \sum_{n=0}^{\infty} \frac{e^{2\pi i n a}}{|n + c|^s}.$$

This extended function analytically continues in the s -variable to the s -plane, with a suitable functional equation. We note three features of this extension:

- (1) The real-analytic extension obeys *twisted-periodicity conditions* in the a and c variables:

$$\begin{aligned}\zeta_*(s, a + 1, c) &= \zeta_*(s, a, c), \\ \zeta_*(s, a, c + 1) &= e^{-2\pi i a} \zeta_*(s, a, c),\end{aligned}$$

- (2) The extended function satisfies two symmetrized four-term functional equations. This fact for $0 < a < 1, 0 < c < 1$ was originally noted by A. Weil [39, pp. 54–58]. These functional equations generalize that of the Riemann zeta function and are given in Sect. 5.1.
- (3) The function $\zeta_*(s, a, c)$ has discontinuities in the a and c variables at integer values of a, c , for various ranges of s . These discontinuities encode information about the nature of the singularities of these functions in the (a, c) -variables at integer values.

The extended Lerch function $\zeta_*(s, a, c)$ will be our extension of the Lerch function to $\mathbb{R} \times \mathbb{R}$; it is well-defined off of grid of horizontal and vertical lines.

The function spaces on $\mathbb{R} \times \mathbb{R}$ that we consider will be suitable classes of (almost everywhere defined) functions $F(a, c)$ that satisfy the twisted-periodicity conditions above:

$$\begin{aligned}F(a + 1, c) &= F(a, c), \\ F(a, c + 1) &= e^{-2\pi i a} F(a, c).\end{aligned}$$

The values of such functions are completely determined by their values in the unit square \square in the (a, c) -variables. A key feature is that the two-variable Hecke operators preserve the twisted-periodicity property. In consequence, it suffices to study these functions inside the unit square, since twisted-periodicity then uniquely extends the function to $\mathbb{R} \times \mathbb{R}$, providing a means of defining the two-variable Hecke operator action on $\mathbb{R} \times \mathbb{R}$ using only function values defined inside the unit square.

Another important ingredient of our extension is an operator encoding the functional equations. We can reframe the functional equation in terms of a new operator acting on functions on $\mathbb{R} \times \mathbb{R}$, the R-operator defined by

$$R(F)(a, c) := e^{-2\pi i a c} F(1 - c, a). \quad (9)$$

This operator has order 4 and it takes the space of continuous functions defined on the unit square \square into itself and extends to define an isometry of the function space $L^p(\square, da dc)$ for $1 \leq p \leq \infty$. Furthermore, when viewed as acting on functions with domain $\mathbb{R} \times \mathbb{R}$, it preserves the subspace of twisted-periodic functions.

Our results incorporating the functional equation will be invariant with respect to the action of the R-operator acting on suitable function spaces. The action of this operator

¹There is no need to take the absolute value in this formula, but it becomes useful in further generalizations.

leads to four families of two-variable Hecke operators that take the following form. These families are

$$T_m, S_m := RT_mR^{-1}, T_m^\vee := R^2T_mR^{-2}, \text{ and } S_m^\vee := R^3T_mR^{-3},$$

explicitly given as

$$\begin{aligned} T_m f(a, c) &= \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{a+k}{m}, mc\right), \\ S_m f(a, c) &= \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i k a} f\left(ma, \frac{c+k}{m}\right), \\ T_m^\vee f(a, c) &= \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i \left(\frac{1-m}{m}a+k\right)} f\left(\frac{a+k}{m}, 1+m(c-1)\right), \\ S_m^\vee f(a, c) &= \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i (m-(k+1))a} f\left(1+m(a-1), \frac{c+m-(k+1)}{m}\right). \end{aligned}$$

Each of these families of operators leaves invariant one side of the unit square in (a, c) -coordinates, and between them they account for all four sides of the unit square. The members of each family commute with other members of the same family. A more surprising result is that, in the function spaces we consider below, members of different families of these operators also commute.

To obtain results that apply to the four families and for all complex s , we introduce function spaces of piecewise continuous functions, allowing discontinuities. The restriction to piecewise continuous functions, without completing the function space, is made because the singularities of the Lerch zeta function at the boundary of the unit square become large as $\Re(s) \rightarrow -\infty$. For values of s inside the critical strip $0 < \Re(s) < 1$, we are able to work inside various Banach spaces. Of particular interest is the Banach space $L^1(\square, da dc)$, which is relevant since the functions $L^\pm(s, a, c)$ belong to this space inside the critical strip ([28, Theorem 2.4]). More generally, one may consider the spaces $L^p(\square, da dc)$ for $1 \leq p \leq 2$. Note that for twisted-periodic functions $F(a, c)$ the L^p -norm of $F(a, c)$ is invariant under measurement in any unit square $x_0 \leq a \leq x_0 + 1, y_0 \leq c \leq y_0 + 1$, allowing real values x_0, y_0 .

The Hilbert space $\mathcal{H} = L^2(\square, da dc)$ is also of great interest; however, the Lerch zeta function $\zeta(s, a, c)$ for $(a, c) \in \square$ does not belong to this space for any $s \in \mathbb{C}$; for $\Re(s) > 1$, this follows from the single term c^{-s} not being in \mathcal{H} , while for other s it is more subtle, see the proof of [28, Theorem 2.4]. The operators R and the four families of Hecke operators induce well-defined actions of bounded operators on this space and together generate an interesting noncommutative algebra of operators that is a \star -algebra in $\mathcal{B}(\mathcal{H})$ which seems worthy of further study.

To obtain function spaces that allow an action of differential operators, we restrict to suitable smaller spaces of piecewise smooth functions that allow discontinuities located at a fixed lattice of axis-parallel vertical lines and horizontal lines, with relevant coordinate vector (horizontal and/or vertical) contained in a lattice $\frac{1}{d}\mathbb{Z}$ for a fixed d depending on the function. We can start this construction with such functions defined on the unit square with specified discontinuities produced by twisted-periodicity having relative coordinate vectors in the same lattice $\frac{1}{d}\mathbb{Z}$.

1.2 Motivation and results

One motivation for this study concerns finding an operator extension of the Riemann zeta function. We may formally define a *zeta operator* Z acting on the domain \mathcal{D}_{++} as

$$Z := \sum_{m=1}^{\infty} T_m.$$

Now one can check that for a fixed complex value s with $\Re(s) > 1$ this operator has the Lerch zeta function $\zeta(s, a, c)$ as an eigenfunction and the Riemann zeta value $\zeta(s)$ as an eigenvalue, that is,

$$Z(\zeta)(s, a, c) = \zeta(s)\zeta(s, a, c).$$

We do not know how to make direct sense of the operator Z in a larger domain in the s -parameter, but the problem to do so provides a motivation for extending the action of the individual two-variable Hecke operators T_m to suitable function spaces, which this paper addresses. In the process, we obtain an extension inside $L^1(\square, da dc)$ to the parameter range $0 < \Re(s) < 1$ in which $T_m(\zeta(s, a, c)) = m^{-s}\zeta(s, a, c)$ holds for all individual Hecke operators T_m .

Precise statements of the main results are given in Sect. 2, but we first make some general remarks.

- (1) We introduce a set of auxiliary operators acting on twisted-periodic function spaces, a linear partial differential operator D_L , a unitary operator R and the family of two-variable Hecke operators T_m , viewed as acting on the unit square \square through the use of twisted-periodic function spaces. We determine commutation relations between all these operators on these function spaces.
- (2) For each $s \in \mathbb{C}$, we define a two-dimensional vector space \mathcal{E}_s , the *Lerch eigenspace*, consisting of twisted-periodic real-analytic functions defined on $\mathbb{R}^2 \setminus \mathbb{Z}^2$ (but sometimes discontinuous on the grid $(\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z})$) which satisfy the eigenvalue identity

$$T_m(\zeta_*)(s, a, c) = m^{-s}\zeta_*(s, a, c) \tag{10}$$

simultaneously for all $m \geq 1$. The spaces \mathcal{E}_s are preserved by the R -operator, and those at s and $1 - s$ are related under the symmetries of the functional equation.

- (3) We give a simultaneous Hecke eigenfunction interpretation of the Lerch zeta function, in the spirit of Milnor's [32] characterization of Kubert functions, including the Hurwitz zeta function. In Sect. 6, we show that for each $s \in \mathbb{C}$ there is a two-dimensional vector space \mathcal{E}_s of simultaneous eigenfunctions, the Lerch eigenspace, satisfying suitable integrability side conditions. This is a generalization of Milnor's converse result characterizing the Hurwitz zeta function and Kubert functions.

In the concluding section, we discuss the possible relevance to the Riemann hypothesis of the structures associated with the Lerch zeta function studied in this series of papers. Many previous formulations of the Riemann hypothesis have analogs in terms of properties of the Lerch zeta function. The operator D_L has features suggested for a putative "Hilbert–Polya" operator.

1.3 Related work

We have already mentioned the definition of two-variable Hecke operators in 1989 by Sun [37] in connection with covering systems, as well as the work of Porubsky [34] noting the eigenfunction property (7) of the Lerch zeta function. See also a survey paper of Porubsky and Schönheim [35] on Erdős's work on covering systems.

Related operators in one variable appeared earlier in a number of different contexts. On specializing to functions $f(a, c)$ that are constant in the c -variable, we obtain one-variable difference operators of the form

$$T_m(f)(a) := \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{a+k}{m}\right). \quad (11)$$

This operator coincides for $m = q$ with an operator U_q^* introduced in 1970 by Atkin and Lehner [3, p. 138], which they termed a *Hecke operator*. They motivate their choice of name "Hecke operator" to 1957 work of Wohlfahrt [40], and we follow their terminology in naming the two-variable operators (6). These operators also appear, denoted U_m , in the modular forms literature as an action on q -expansions, as $U_m(\sum_{n=0}^{\infty} a_n q^n) := \sum_{n=0}^{\infty} a_{mn} q^n$. If now $f = \sum_{n=0}^{\infty} a_n q^n$, then under the change in variable $q = e^{2\pi i \tau}$ with τ in the upper half-plane, this action becomes $U_m(f)(\tau) = \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{\tau+k}{m}\right)$, see Lang [31, p. 108], Koblitz [22, pp. 161-2]. In an analogous p -adic modular form context the operator U_p is sometimes called *Atkin's operator*, see Dwork [18] and Katz [21].

In 1983, Milnor [32] considered operators of form (11), calling them *Kubert operators*. General operators of the type (11) acting on an abelian group (e.g., \mathbb{R}/\mathbb{Z}) had previously been studied by Kubert and Lang [24] and Kubert [23]. Milnor characterized simultaneous eigenfunction solutions of such operators, in the space of continuous functions on the open interval $(0, 1)$. In Sect. 6.1, we review Milnor's work.

On specializing to functions $f(a, c)$ that are constant in the a -variable, we obtain a different family of one-variable operators, the *dilation operators*

$$\tilde{T}_m(f)(c) := f(mc). \quad (12)$$

These operators satisfy the identities

$$\tilde{T}_m \circ \tilde{T}_n = \tilde{T}_n \circ \tilde{T}_m = \tilde{T}_{mn}.$$

for all $m, n \geq 1$. The dilation operator \tilde{T}_m coincides in form with an operator A_m introduced in 1970 in Atkin and Lehner [3, equation (2.1)].² The dilation operators also appear, denoted V_m , in the modular forms literature, acting on q -expansions as $V_m(\sum_{n=0}^{\infty} a_n q^n) = \sum_{n=0}^{\infty} a_n q^{mn}$. If now $f = \sum_{n=0}^{\infty} a_n q^n$, then under the change in variable $q = e^{2\pi i \tau}$ this action becomes $V_m(f)(\tau) = f(m\tau)$, see Lang [31, p. 108], Koblitz [22, pp. 161–162]. In 1945, Beurling [10] studied completeness properties of dilates $\{\tilde{T}_m f : m \geq 1\}$ viewed in $L^2(0, 1)$ of a periodic function f on \mathbb{R} of period 1.

In 1999, Báez-Duarte [4] noted that this family of dilation operators relates to the real-variables approach to the Riemann hypothesis due to Nyman [33] and Beurling [11], see also Báez-Duarte [4] and Burnol [14, 15]. Convergence of series composed out of dilations of functions has been much studied; they include Fourier series as a special case, see Gaposkin [19], Aisletiner et al. [1, 2], Berkes and Weber [7] and Weber [38].

² The operator A_n appears for prime p as $A_p = T_p^* - U_p^*$. However, Atkin and Lehner apply T_p^* only for $p \nmid N$, the level, and U_p^* for $p|N$, but the definitions of each operator make sense for all p , so we may relate them in the identity above.

2 Main results

In this paper, we study actions of the two-variable Hecke operators on different function spaces and determine their commutation relations relative to various other operators appearing in parts I–III.

2.1 Two-variable Hecke operators and R-operator

In Sect. 3, we introduce and study functional operators and differential operators on functions on the open unit square $\square^\circ = (0, 1)^2$. The first of these operators is the R-operator defined by

$$R(F)(a, c) = e^{-2\pi iac} F(1 - c, a).$$

As noted earlier, this operator satisfies $R^4 = I$; it plays a role analogous to the Fourier transform. The functional equations for the Lerch zeta function given in (4) and (5) can be put in an elegant form using the R-operator, see Proposition 2.

The second operator D_L is constructed using the linear partial differential operators $D_L^+ = \frac{\partial}{\partial c}$ and $D_L^- = \frac{1}{2\pi i} \frac{\partial}{\partial a} + c$ for which one has

$$\begin{aligned} (D_L^+ \zeta)(s, a, c) &= -s\zeta(s + 1, a, c) \\ (D_L^- \zeta)(s, a, c) &= \zeta(s - 1, a, c). \end{aligned} \quad (13)$$

These operators define unbounded operators on $L^2(\square, da dc)$ and on $L^1(\square, da dc)$. In the remainder of Sect. 3, we determine commutation relations between these operators on the space $C^{1,1}(\square^\circ)$ of jointly continuously differentiable functions on the open unit square \square° whose mixed second partials are continuous and on which the two first partials commute.

In Sect. 4, we construct the “real-variables” action of the two-variable Hecke operators on an extended function space. We also consider the Hecke operators conjugated by powers of R:

$$S_m := RT_mR^{-1}, \quad T_m^\vee := R^2T_mR^{-2}, \quad \text{and} \quad S_m^\vee := R^3T_mR^{-3}.$$

Together with T_m , we have four families of operators, each of which formally leaves invariant a line passing through one side of the square \square .

Theorem 1 (Commuting Operator Families)

- (1) *The four sets of two-variable Hecke operators $\{T_m, S_m, T_m^\vee, S_m^\vee : m \geq 1\}$ continuously extend to bounded operators on each Banach space $L^p(\square, da dc)$ for $1 \leq p \leq \infty$, by viewing these (almost everywhere defined) functions of \square as extended to $\mathbb{R} \times \mathbb{R}$ via the twisted-periodicity relations $f(a + 1, c) = f(a, c)$ and $f(a, c + 1) = e^{-2\pi ia} f(a, c)$. These operators satisfy $T_m = T_m^\vee$, $S_m = S_m^\vee$ and $S_m = \frac{1}{m}(T_m)^{-1}$ for all $m \geq 1$.*
- (2) *The \mathbb{C} -algebra \mathcal{A}_0^p of operators on $L^p(\square, da dc)$ generated by these four sets of operators under addition and operator multiplication is commutative.*
- (3) *On $L^2(\square, da dc)$ the adjoint Hecke operator $(T_m)^* = S_m$, and $(S_m)^* = T_m$. In particular, the \mathbb{C} -algebra \mathcal{A}_0^2 is a \star -algebra. In addition each of $\sqrt{m}T_m$, $\sqrt{m}S_m$ is a unitary operator on $L^2(\square, da dc)$.*

For all $p \geq 1$, the operator R defines an isometry $\|R(f)\|_p = \|f\|_p$ on $L^p(\square, da dc)$. In consequence, we obtain an extended algebra $\mathcal{A}^p := \mathcal{A}_0^p[R]$ of operators, by adjoining the operator R to \mathcal{A}_0^p . The algebra \mathcal{A}^p is a noncommutative algebra. The algebra \mathcal{A}^2 is also a \star -algebra, which follows using $(R)^* = R^{-1} = R^3$.

2.2 Two-dimensional Lerch eigenspace

In Sect. 5, we study the Lerch eigenspace \mathcal{E}_s defined as the vector space over \mathbb{C} spanned by the four functions

$$\mathcal{E}_s := \langle L^+(s, a, c), L^-(s, a, c), e^{-2\pi iac}L^+(1 - s, 1 - c, a), e^{-2\pi iac}L^-(1 - s, 1 - c, a) \rangle,$$

viewing the (a, c) -variables on \square° . We treat the variable s as constant, and all $s \in \mathbb{C}$ are allowed since on \square° the functions $L^\pm(s, a, c)$ are entire functions of s . Note that the gamma factors are omitted from the functions in this definition. These functions satisfy linear dependencies by virtue of the two functional equations that $L^\pm(s, a, c)$ satisfy (Theorem 5).

Theorem 2 (Operators on Lerch eigenspaces) *For each $s \in \mathbb{C}$, the space \mathcal{E}_s is a two-dimensional vector space. All functions in \mathcal{E}_s have the following properties.*

- (i) (Lerch differential operator eigenfunctions) *Each $f \in \mathcal{E}_s$ is an eigenfunction of the Lerch differential operator $D_L = \frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + c \frac{\partial}{\partial c}$ with eigenvalue $-s$, namely*

$$(D_L f)(s, a, c) = -sf(s, a, c)$$

holds at all $(a, c) \in \mathbb{R} \times \mathbb{R}$, with both a and c nonintegers.

- (ii) (Simultaneous Hecke operator eigenfunctions) *Each $f \in \mathcal{E}_s$ is a simultaneous eigenfunction with eigenvalue m^{-s} of all two-variable Hecke operators*

$$T_m(f)(a, c) = \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{a+k}{m}, mc\right)$$

in the sense that, for each $m \geq 1$,

$$T_m f = m^{-s} f$$

holds on the domain $(\mathbb{R} \setminus \frac{1}{m}\mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z})$.

- (iii) (J-operator eigenfunctions) *The space \mathcal{E}_s admits the involution*

$$Jf(a, c) := e^{-2\pi iac} f(1 - a, 1 - c),$$

under which it decomposes into one-dimensional eigenspaces $\mathcal{E}_s = \mathcal{E}_s^+ \oplus \mathcal{E}_s^-$ with eigenvalues ± 1 , that is, $\mathcal{E}_s^\pm = \langle F_s^\pm \rangle$ and

$$J(F_s^\pm) = \pm F_s^\pm.$$

- (iv) (R-operator action) *The R-operator $R(F)(a, c) = e^{-2\pi iac} F(1 - c, a)$ acts by $R(\mathcal{E}_s) = \mathcal{E}_{1-s}$ with*

$$R(L^\pm(s, a, c)) = w_\pm^{-1} \gamma^\pm(1 - s) L^\pm(1 - s, a, c),$$

where $w_+ = 1, w_- = i, \gamma^+(s) = \Gamma_{\mathbb{R}}(s)/\Gamma_{\mathbb{R}}(1 - s), \gamma^-(s) = \gamma^+(s + 1)$ and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$.

The members of the Lerch eigenspaces are shown to have the following analytic properties (Theorem 6).

Theorem 3 (Analytic Properties of Lerch eigenspaces) *For fixed $s \in \mathbb{C}$, the functions in the Lerch eigenspace \mathcal{E}_s are real-analytic functions of (a, c) on $(\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z})$, which may be discontinuous at values $a, c \in \mathbb{Z}$. They have the following properties.*

(i) (Twisted-Periodicity Property) All functions $F(a, c)$ in \mathcal{E}_s satisfy the twisted-periodicity functional equations

$$F(a + 1, c) = F(a, c),$$

$$F(a, c + 1) = e^{-2\pi ia} F(a, c).$$

(ii) (Integrability Properties)

(a) If $\Re(s) > 0$, then for each noninteger c all functions in \mathcal{E}_s have $f_c(a) := F(a, c) \in L^1[(0, 1), da]$, and all their Fourier coefficients

$$f_n(c) := \int_0^1 F(a, c) e^{-2\pi ina} da, \quad n \in \mathbb{Z},$$

are continuous functions of c on $0 < c < 1$.

(b) If $\Re(s) < 1$, then for each noninteger a all functions in \mathcal{E}_s have $g_a(c) := e^{2\pi iac} F(a, c) \in L^1[(0, 1), dc]$, and all Fourier coefficients

$$g_n(a) := \int_0^1 e^{2\pi iac} F(a, c) e^{-2\pi inc} dc, \quad n \in \mathbb{Z},$$

are continuous functions of a on $0 < a < 1$.

(c) If $0 < \Re(s) < 1$, then all functions in \mathcal{E}_s belong to $L^1[\square, dadc]$. In this range, the vector space \mathcal{E}_s is invariant under the action of all four sets $T_m, S_m, T_m^\vee, S_m^\vee$ ($m \geq 1$) of two-variable Hecke operators.

2.3 Eigenfunction characterization of Lerch eigenspace

In Sect. 6, we first review Milnor’s simultaneous eigenfunction characterization of Kubert functions (Theorem 7). We then prove the following characterization for the Lerch eigenspace \mathcal{E}_s (Theorem 8), the main result of this paper, which can be viewed as a generalization to two dimensions of Milnor’s result. It may also be regarded as a converse theorem to Theorem 3.

Theorem 4 (Lerch Eigenspace Characterization) *Let $s \in \mathbb{C}$. Suppose that $F(a, c) : (\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z}) \rightarrow \mathbb{C}$ is a continuous function that satisfies the following conditions.*

(1) (Twisted-Periodicity Condition) For $(a, c) \in (\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z})$,

$$F(a + 1, c) = F(a, c)$$

$$F(a, c + 1) = e^{-2\pi ia} F(a, c).$$

(2) (Integrability Condition) At least one of the following two conditions (2-a) or (2-c) holds. (2-a) The s -variable has $\Re(s) > 0$. For $0 < c < 1$ each function $f_c(a) := F(a, c) \in L^1[(0, 1), da]$, and all the Fourier coefficients

$$f_n(c) := \int_0^1 f_c(a) e^{-2\pi ina} da = \int_0^1 F(a, c) e^{-2\pi ina} da, \quad n \in \mathbb{Z},$$

are continuous functions of c .

(2-c) The s -variable has $\Re(s) < 1$. For $0 < a < 1$ each function $g_a(c) := e^{2\pi iac} F(a, c) \in L^1[(0, 1), dc]$, and all the Fourier coefficients

$$g_n(a) := \int_0^1 g_a(c)e^{-2\pi inc} dc = \int_0^1 e^{2\pi iac} F(a, c)e^{-2\pi inc} dc, \quad n \in \mathbb{Z},$$

are continuous functions of a .

(3) (Hecke Eigenfunction Condition) For all $m \geq 1$,

$$T_m(F)(a, c) = m^{-s}F(a, c)$$

holds for $(a, c) \in (\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \frac{1}{m}\mathbb{Z})$.

Then $F(a, c)$ is the restriction to noninteger (a, c) -values of a function in the Lerch eigenspace \mathcal{E}_s .

This generalization imposes extra integrability conditions in order to control the dilation property of the two-variable Hecke operators in the c -coordinate. The twisted-periodicity hypothesis plays an essential role in the proof of Theorem 4, in giving identities that the Fourier series coefficients of such functions must satisfy, see (84).

2.4 Further extensions

One may further define generalizations of the Lerch zeta function that incorporate Dirichlet characters $\chi \pmod{N}$. In the real-analytic case, they take the form

$$L^\pm(\chi, s, a, c) = \sum_{n \in \mathbb{Z}} \chi(n)(\text{sgn}(n + c))^k e^{2\pi ina} |n + c|^{-s},$$

with $k = 0, 1$. These functions can be expressed as linear combinations of scaled versions of $L^\pm(s, a, c)$. The Hecke operator framework extends to apply to such functions, with some extra complications, see [27].

Subsequent work of the first author [27] gives a representation-theoretic interpretation of the “real-variables” version of the Lerch zeta function treated in this paper, related to function spaces associated with the Heisenberg group. Under it, various combinations of Lerch zeta function and variants twisted by Dirichlet characters are found to play the role of Eisenstein series with respect to the operator $\Delta_L = D_L + \frac{1}{2}\mathbf{I}$, where \mathbf{I} is the identity operator, which plays the role of a Laplacian. The Lerch zeta functions treated in this paper correspond to the trivial Dirichlet character. In this Eisenstein series interpretation, the Lerch functions $L^\pm(s, a, c)$ on the line $\Re(s) = \frac{1}{2}$ parametrize a pure continuous spectrum of the operator Δ_L on a suitable Hilbert space \mathcal{H} . The real-analytic Hecke operators T_m described in this paper then correspond to dilation operators similar to \tilde{T}_m in Sect. 1.3.

In another sequel paper, the first author studies a “complex variables” two-variable Hecke operator action associated to the Lerch zeta functions. This framework again includes the Lerch zeta function for $\Re(s) > 1$, with two-variable Hecke operators initially viewed as acting on a suitable domain of holomorphic functions. It then studies these Hecke operators acting on spaces of (multivalued) holomorphic functions on various domains. These are quite different function spaces than the ones treated here and the resulting Hecke action has new features. In particular, members of different families of Hecke operators do not commute on these function spaces.

Notation Much of the analysis of this paper concerns functions with $s \in \mathbb{C}$ regarded as a parameter. We let $\square = \{(a, c) \in [0, 1] \times [0, 1]\}$, and its interior $\square^\circ = \{(a, c) \in (0, 1) \times (0, 1)\}$.

3 R-operator and differential operators

We introduce various operators with respect to which the Lerch zeta function has invariance properties. We first restate operators acting on functions of two real variables (a, c) defined almost everywhere on the unit square \square° . Then we consider operators defined on larger domains of $\mathbb{R} \times \mathbb{R}$. We start with a nonlocal operator R mentioned before, associated with the Fourier transform, and certain linear partial differential operators, which we term real-analytic Lerch differential operators. In Sect. 3.1, we define the R -operator and reformulate the functional equations for the Lerch zeta function in part I, using this operator. In Sect. 3.2, we define Lerch differential operators, and in Sect. 3.3, we determine their commutation relations on suitable function spaces, together with the R -operator.

3.1 Local operators: Real-variable differential operators

We first study transformation of the Lerch zeta function under certain (real variable) linear partial differential operators. Recall from Sect. 2.1 the operators

$$D_L^+ = \frac{\partial}{\partial c}, \quad (14)$$

called the “raising operator,” and

$$D_L^- = \frac{1}{2\pi i} \frac{\partial}{\partial a} + c \quad (15)$$

called the “lowering operator.” The names are suggested by the property that for $\Re(s) > 1$ the Lerch zeta function satisfies the recurrence equations

$$(D_L^+ \zeta)(s, a, c) = -s\zeta(s + 1, a, c) \quad (16)$$

and

$$(D_L^- \zeta)(s, a, c) = \zeta(s - 1, a, c), \quad (17)$$

which raise and lower the value of the s parameter. These identities follow using its series definition (1).

We combine the two operators to obtain the *Lerch differential operator* D_L as in (2), given by

$$D_L = D_L^- D_L^+ = \left(\frac{1}{2\pi i} \frac{\partial}{\partial a} + c \right) \frac{\partial}{\partial c}. \quad (18)$$

It follows from (16) and (17) that the Lerch zeta function is an eigenfunction of the Lerch differential operator for $\Re(s) > 1$, satisfying

$$(D_L \zeta)(s, a, c) = -s\zeta(s, a, c), \quad (19)$$

as noted in (3).

Lemma 1 *The operator D_L takes a piecewise $C^{1,1}$ -function that is twisted-periodic on $\mathbb{R} \times \mathbb{R}$ to a twisted-periodic function on $\mathbb{R} \times \mathbb{R}$.*

Proof For piecewise $C^{1,1}$ -functions, the derivatives $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial c}$ commute. (We allow discontinuities at the edges of the pieces, with the boundaries being smooth curves; straight line boundaries are used in the sequel.) By definition, a twisted-periodic function F on $\mathbb{R} \times \mathbb{R}$ satisfies $F(a + 1, c) = F(a, c)$ and $F(a, c + 1) = e^{-2\pi i a} F(a, c)$. We have to show these two relations with F replaced by $D_L F$, whenever partial derivatives are allowed. Here $(D_L F)(a, c)$ means

$$(D_L F)(a, c) = \left(\frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + c \frac{\partial}{\partial c} \right) (F(a, c)).$$

Note that $\frac{\partial}{\partial a} = \frac{\partial}{\partial(a+1)}$ and $\frac{\partial}{\partial c} = \frac{\partial}{\partial(c+1)}$ by the chain rule. It is easy to check that $(D_L F)(a + 1, c) = (D_L F)(a, c)$. For the remaining relation, using twisted-periodicity gives

$$\begin{aligned} (D_L F)(a, c + 1) &= \left(\frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + (c + 1) \frac{\partial}{\partial c} \right) (F(a, c + 1)) \\ &= \left(\frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + (c + 1) \frac{\partial}{\partial c} \right) (e^{-2\pi i a} F(a, c)) \\ &= e^{-2\pi i a} \frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} (F(a, c)) + e^{-2\pi i a} (-1 + c + 1) \frac{\partial}{\partial c} (F(a, c)) \\ &= e^{-2\pi i a} (D_L F)(a, c), \end{aligned}$$

as required. □

3.2 Nonlocal operator: R-operator

Recall from (9) the *R-operator* which acts on suitable functions on the open square \square° given by

$$R(f)(a, c) = e^{-2\pi i a c} f(1 - c, a). \tag{20}$$

This operator satisfies $R^4 = I$, with

$$R^2(f)(a, c) = e^{-2\pi i a} f(1 - a, 1 - c), \tag{21}$$

$$R^3(f)(a, c) = e^{-2\pi i a c + 2\pi i c} f(c, 1 - a), \tag{22}$$

$$R^4(f)(a, c) = f(a, c). \tag{23}$$

Note that the operator J in Theorem 2 is nothing but R^2 , which we sometimes term the *reflection operator* since it relates the function values in each coordinate about the point $a = \frac{1}{2}$, resp. $c = \frac{1}{2}$.

These operators have well-defined actions on continuous functions $C^0((0, 1)^2)$ defined in the open unit square. They extend by closure to an operator action on (almost everywhere defined) L^p -functions on the unit square, for any $p \geq 1$. In particular, the extended action on $L^2(\square, dadc)$ is unitary.

Lemma 2 *The operator R preserves the property of being (almost everywhere) twisted-periodic on $\mathbb{R} \times \mathbb{R}$.*

Proof Given a twisted-periodic $F(a, c)$ on $\mathbb{R} \times \mathbb{R}$, set $g(a, c) = R(F)(a, c) = e^{-2\pi i a c} F(1 - c, a)$. Then

$$\begin{aligned} g(a, c + 1) &= e^{-2\pi i a(c+1)} F(1 - (c + 1), a) \\ &= e^{-2\pi i a c} e^{-2\pi i a} F(-c, a) \\ &= e^{-2\pi i a} (e^{-2\pi i a c} F(1 - c, a)) = e^{-2\pi i a} g(a, c) \end{aligned}$$

where twisted-periodicity of F was used on the third line. Then

$$\begin{aligned} g(a + 1, c) &= e^{-2\pi i(a+1)c} F(1 - c, a + 1) \\ &= e^{-2\pi i a c} e^{-2\pi i c} F(1 - c, a + 1) \\ &= e^{-2\pi i a c} e^{-2\pi i c} (e^{2\pi i c} F(1 - c, a)) = g(a, c), \end{aligned}$$

where twisted-periodicity of F was used on the third line.

Remark 1 The R-operator can be extended to act pointwise on functions defined almost everywhere on \square° , with the same rule (20). That is, we can allow singularities on a finite set of horizontal and vertical lines. It can also be extended to the domain $\mathbb{R} \times \mathbb{R}$, with the same rule (20), acting on functions satisfying a twisted-periodic relation, see Lemma 4.

3.3 Commutation relations

Let $C^{1,1}(\square^\circ)$ denote the complex vector space of jointly continuously differentiable functions $f(a, c)$ on the open unit square \square° whose mixed second partials are continuous functions and satisfy

$$\frac{\partial^2 f}{\partial a \partial c}(a, c) = \frac{\partial^2 f}{\partial c \partial a}(a, c).$$

The following result describes commutation relations of these operators among themselves and with the R-operator.

Lemma 3 *On the space $C^{1,1}(\square^\circ)$ the following properties hold.*

- (i) *The operators $D_L^+ = \frac{\partial}{\partial c}$ and $D_L^- = \frac{1}{2\pi i} \frac{\partial}{\partial a} + c$ satisfy the commutation relation*

$$D_L^+ D_L^- - D_L^- D_L^+ = I, \tag{24}$$

where I denotes the identity operator.

- (ii) *The operator $Rf(a, c) := e^{-2\pi iac} f(1 - c, a)$ leaves $C^{1,1}(\square^\circ)$ invariant and satisfies*

$$D_L^+ R = -2\pi i R D_L^-, \tag{25}$$

$$D_L^- R = \frac{1}{2\pi i} R D_L^+. \tag{26}$$

- (iii) *The operator $D_L = D_L^- D_L^+$ satisfies*

$$D_L R + R D_L = -R. \tag{27}$$

In particular, D_L and R^2 commute:

$$D_L R^2 = R^2 D_L. \tag{28}$$

Proof These results follow by direct calculation. For (i), we have

$$D_L^+ D_L^- = \frac{\partial}{\partial c} \left(\frac{1}{2\pi i} \frac{\partial}{\partial a} + c \right) = \frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + c \frac{\partial}{\partial c} + I = D_L^- D_L^+ + I.$$

For (ii), we use D_1, D_2 to denote partial derivatives with respect to the first and second variable, respectively, in order to avoid ambiguity under the action of R , which interchanges the variables. Let

$$h(a, c) := (D_L^- f)(a, c) = \left(\frac{1}{2\pi i} D_1 f + cf \right)(a, c).$$

Then

$$\begin{aligned} (R D_L^- f)(a, c) &= (R h)(a, c) = e^{-2\pi iac} h(1 - c, a) \\ &= e^{-2\pi iac} \left(\frac{1}{2\pi i} (D_1 f)(1 - c, a) + af(1 - c, a) \right). \end{aligned}$$

On the other hand, let

$$g(a, c) = f(1 - c, a)$$

so that $(Rf)(a, c) = e^{-2\pi iac}g(a, c)$. We have

$$\begin{aligned} (D_L^+ Rf)(a, c) &= -2\pi iae^{-2\pi iac}g(a, c) + e^{-2\pi iac}(D_2g)(a, c) \\ &= -2\pi iae^{-2\pi iac}f(1 - c, a) - e^{-2\pi iac}(D_1f)(1 - c, a) \\ &= -2\pi i(RD_L^- f)(a, c). \end{aligned}$$

In other words, $D_L^+ R = -2\pi iRD_L^-$, which is (25).

Next, we have

$$\begin{aligned} (D_L^- Rf)(a, c) &= \frac{1}{2\pi i} \frac{\partial}{\partial a} (Rf)(a, c) + c(Rf)(a, c) \\ &= \frac{1}{2\pi i} (-2\pi ice^{-2\pi iac}f(1 - c, a) + e^{-2\pi iac}D_1g(a, c)) + ce^{-2\pi iac}f(1 - c, a) \\ &= \frac{1}{2\pi i} e^{-2\pi iac}D_2f(1 - c, a) = \frac{1}{2\pi i} (RD_L^+ f)(a, c). \end{aligned}$$

In other words, $D_L^- R = \frac{1}{2\pi i} RD_L^+$, which is (26).

For (iii) we use (i) and (ii) to obtain

$$\begin{aligned} D_L R &= D_L^- D_L^+ R = -2\pi iD_L^- RD_L^- = -RD_L^+ D_L^- \\ &= -R(D_L^- D_L^+ + I) = -RD_L - R, \end{aligned}$$

which is (27). This identity yields

$$D_L R^2 = (-RD_L - R)R = -R(D_L R + R) = -R(-RD_L) = R^2 D_L,$$

which is (28). □

4 Two-variable Hecke operators

The individual terms in the sums defining two-variable Hecke operators (6) are operators that act on the domain by affine changes in variable. They are compositions of a -variable operators of form

$$T_{k,m}(f)(a, c) := f\left(\frac{a+k}{m}, c\right),$$

which map the unit square into itself, together with the c -variable dilation operator

$$D_m^c(f)(a, c) := f(a, mc),$$

which maps the unit square outside itself. A main problem in obtaining a well-defined action of the two-variable Hecke operators (6) is to accommodate the “dilation” action in the c -variable, which expands the domain. In the c variable, the domain must therefore include at least $\mathbb{R}_{>0}$ or $\mathbb{R}_{<0}$. The additive action in the a -variable requires to handle all $T_{k,m}$ for $0 \leq k < m$ a “nonlocal” definition valid over an interval of width at least one.

The viewpoint of this paper is start with functions defined (almost everywhere) on the unit square $\square = [0, 1] \times [0, 1]$, extend them to functions defined on the plane $\mathbb{R} \times \mathbb{R}$ by imposing twisted-periodicity conditions: $f(a+1, c) = f(a, c)$ and $f(a, c+1) = e^{-2\pi ia}f(a, c)$, and use the extended functions to define an action of the two-variable Hecke operators on functions on the unit square; the resulting action is linear and preserves twisted-periodicity conditions on the functions.

We would like to accommodate functions with discontinuities of the type that naturally appear in connection with the Lerch zeta function in the real-analytic framework, which occur for integer values of a and/or c . We must also deal with the problem that the

action of T_m changes the location of the discontinuities. We will introduce appropriate function spaces that are closed under the Hecke operator action allowing a finite (but variable) number of discontinuities in horizontal and vertical directions. These operators also make sense on the Banach space $L^1(\square, da dc)$ and on the Hilbert space $L^2(\square, da dc)$.

4.1 Twisted-periodic function spaces

In the real-variables framework treated in this paper, we take as a function space the set of piecewise continuous functions with a finite number of pieces defined in the open unit square \square° , allowing discontinuities on horizontal and vertical lines and permitting the number of discontinuities to depend on the function.

Definition 1 (i) The *twisted-periodic function space* $\mathcal{P}(\square^\circ)$ is the complex vector space that consists of equivalence classes of functions $f : \square^\circ \rightarrow \mathbb{C}$ which are piecewise continuous, with discontinuities, if any, located along a finite number of horizontal lines at rational coordinates $c = \frac{j}{d}$, $0 < j < d$, for some integer $d \geq 1$, called an *admissible denominator* for the function. We then extend these functions to functions $f : (\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z})$, by imposing the twisted-periodicity conditions

$$f(a + 1, c) = f(a, c) \tag{29}$$

$$f(a, c + 1) = e^{-2\pi ia} f(a, c) \tag{30}$$

and denote the set of such equivalence classes of functions $\mathcal{P}(\mathbb{R} \times \mathbb{R})$. Here two functions with admissible denominators d, d' are considered *equivalent*, if their values coincide on the set

$$S_{dd'} := \{(a, c) \in (\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \frac{1}{dd'}\mathbb{Z})\}.$$

(ii) The smallest lattice $\frac{1}{d}\mathbb{Z}$ on which discontinuities occur (for $d \geq 1$) will be called the *support lattice* of the function, and the minimal value d can be called the *conductor* of the function.

We allow “functions” in $\mathcal{P}(\mathbb{R} \times \mathbb{R})$ to be undefined on the set of horizontal lines with coordinates $c = \frac{j}{d}$ for all $j \in \mathbb{Z}$ and do not compare their values at such points. Note that a function having admissible denominator d can be regarded also a function with admissible denominator kd for any integer $k \geq 1$. Aside from this ambiguity, any function in $\mathcal{P}(\mathbb{R} \times \mathbb{R})$ is completely determined by its values in $\mathcal{P}(\square^\circ)$, using the twisted-periodicity conditions. More precisely, the equivalence relation on “functions” in $\mathcal{P}(\mathbb{R} \times \mathbb{R})$ calls functions equivalent whose values agree outside of a set of measure zero; the space $\mathcal{P}(\mathbb{R} \times \mathbb{R})$ has a well-defined vector space structure on equivalence classes.

Proposition 1 (Real-variable Hecke operators) *The two-variable Hecke operators*

$$T_m(f)(a, c) = \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{a+k}{m}, mc\right)$$

act on twisted-periodic functions in $\mathcal{P}(\mathbb{R} \times \mathbb{R})$, i.e., they preserve twisted-periodicity. On this function space, they satisfy

$$T_m T_n = T_n T_m = T_{mn}. \tag{31}$$

Proof If $f(a, c) \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$, with admissible denominator d , then each $f(\frac{a+k}{m}, mc) \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$ with admissible denominator md , hence $T_m(f) \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$ with admissible denominator md .

It remains to verify (31). Assuming that $f(a, c)$ is twisted-periodic with discontinuities at level denominator d , the following calculation is well defined with denominator mnd ,

$$\begin{aligned} T_m(T_n(f))(a, c) &= \frac{1}{m} \sum_{k=0}^{m-1} (T_n(f))\left(\frac{a+k}{m}, mc\right) \\ &= \frac{1}{m} \sum_{k=0}^{m-1} \left(\frac{1}{n} \sum_{l=0}^{n-1} f\left(\frac{\frac{a+k}{m} + l}{n}, n(mc)\right) \right) \\ &= \frac{1}{mn} \sum_{k=0}^{m-1} \left(\sum_{l=0}^{n-1} f\left(\frac{a+k+lm}{mn}, mnc\right) \right) \\ &= T_{mn}(f)(a, c). \end{aligned} \tag{32}$$

The equality $T_n(T_m(f)) = T_{mn}(f)$ follows similarly. □

Remark 2 One may also study two-variable Hecke operators T_m and R acting as bounded operators on the function spaces $L^1(\square, da dc)$ or $L^2(\square, da dc)$. Here, we view the functions as extended to (most of) $\mathbb{R} \times \mathbb{R}$ by twisted-periodicity to define the Hecke operator action. We note that the twisted-periodicity conditions in Definition 1 preserve both the L^1 -norm and the L^2 -norm of functions on unit squares having integer lattice points as corners.

4.2 R-operator conjugates of Hecke operators

Conjugation by powers of R of the Hecke operators gives rise to four related families of two-variable Hecke operators. The first family consists of the operators $\{T_m : m \geq 1\}$ in (6), and the remaining three families are

$$S_m := RT_mR^{-1}, \quad T_m^\vee := R^2T_mR^{-2}, \quad \text{and} \quad S_m^\vee := R^3T_mR^{-3}. \tag{33}$$

Their actions are given by

$$S_m f(a, c) := \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi ika} f\left(ma, \frac{c+k}{m}\right); \tag{34}$$

$$T_m^\vee f(a, c) := \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i\left(\frac{(1-m)a+k}{m}\right)} f\left(\frac{a+k}{m}, 1+m(c-1)\right); \tag{35}$$

$$S_m^\vee f(a, c) := \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i(m-(k+1))a} f\left(1+m(a-1), \frac{c+m-(k+1)}{m}\right). \tag{36}$$

Each of these families of operators leaves invariant a line passing through one side of the square \square , in the sense that the above definitions (formally) make sense as one-variable operators, when restricted to the invariant line. For T_m, T_m^\vee, S_m and S_m^\vee , the invariant lines are $c = 0, c = 1, a = 0$ and $a = 1$, respectively.

In order to have a suitable domain inside $\mathbb{R} \times \mathbb{R}$ on which all four families of operators are simultaneously defined, we must use essentially all of $\mathbb{R} \times \mathbb{R}$, because between them the operators are expansive in the positive and negative a -directions and c -directions. We make the following definition, allowing functions with discontinuities, for the reasons given above.

Definition 2 The *extended twisted-periodic function space* $\mathcal{P}^*(\mathbb{R} \times \mathbb{R})$ is a complex vector space of equivalence classes of functions $f : \square^\circ \rightarrow \mathbb{C}$ which are piecewise continuous, having the property that: there is a positive integer d (which may depend on the function) such that all discontinuities of $f(a, c)$ occur only in the following regions:

- along horizontal lines at rational coordinates $c = \frac{j}{d}, 0 \leq j < d,$
- along vertical lines at rational coordinates $a = \frac{k}{d}, 0 \leq k < d.$

We call any such value d an *admissible denominator* for the function. (The functions need not be well defined along the lines of discontinuity.)

- (i) These functions are extended to functions $f : (\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z}) \rightarrow \mathbb{C},$ by imposing the twisted-periodicity conditions

$$f(a + 1, c) = f(a, c)$$

$$f(a, c + 1) = e^{-2\pi i a} f(a, c).$$

- (ii) Two functions with admissible denominators d, d' are considered *equivalent*, if their values coincide on the set

$$\mathcal{S}_{dd'} := \left\{ (a, c) \in (\mathbb{R} \setminus \frac{1}{dd'}\mathbb{Z}) \times (\mathbb{R} \setminus \frac{1}{dd'}\mathbb{Z}) \right\}.$$

We denote the set of such equivalence classes of functions by $\mathcal{P}^*(\mathbb{R} \times \mathbb{R}).$

- (iii) Given a function in an equivalence class in $\mathcal{P}^*(\mathbb{R} \times \mathbb{R}),$ any integer $d \geq 1$ for which all discontinuities of some function equivalent to $f(a, c)$ are on the lattice $\frac{1}{d}\mathbb{Z}^2$ will be called a *support lattice* of the function. The minimal value d will be called the *conductor* of the function.³

The individual operators $f(a, c) \mapsto \frac{1}{m}f(\frac{a+k}{m}, mc)$ making up the two-variable Hecke operator T_m leave the space $\mathcal{P}^*(\mathbb{R} \times \mathbb{R})$ invariant, and map a function in $\mathcal{P}^*(\mathbb{R} \times \mathbb{R})$ with conductor d to one with conductor dividing $md.$

One can extend the “real-variables” framework to all four of these families of Hecke operators, acting on the full twisted-periodic function space, using conjugation by $R^j.$ This assertion is justified by the following result, in which we regard the operator R as acting on functions with domain values $(a, c) \in \mathbb{R} \times \mathbb{R}$ by (20).

Lemma 4 *The extended twisted-periodic function space $\mathcal{P}^*(\mathbb{R} \times \mathbb{R})$ is invariant under the action of the operator $R.$*

Proof This follows from Lemma 2.

4.3 Commutation relations of two-variable Hecke operators

We now show some surprising commutation relations; on suitable function spaces all four sets of Hecke operators mutually commute, despite their noncommutativity with the R -operator.

³One can define a refined notion which allows discontinuities with denominator d_1 in the c -direction and d_2 in the a -direction. Here d is the least common multiple $d = [d_1, d_2].$ One can also define a refined notation (d_1, d_2) of conductor in the variables separately.

Lemma 5 *Let f be a function in the extended twisted-periodic function vector space $\mathcal{P}^*(\mathbb{R} \times \mathbb{R})$, and let $d \geq 1$ be an integer.*

(1) For $m \geq 1$,

$$S_m \circ T_{dm}(f)(a, c) = \frac{1}{m} T_d(f)(a, c).$$

(2) For $d = 1$ and all $m \geq 1$,

$$S_m \circ T_m(f)(a, c) = T_m \circ S_m(f)(a, c) = \frac{1}{m} f(a, c).$$

Here T_m is invertible and $S_m = \frac{1}{m}(T_m)^{-1}$, and S_m and T_m commute.

Proof The linear operators T_m and S_m take $\mathcal{P}^*(\mathbb{R} \times \mathbb{R})$ into itself. We first let $\ell, m \geq 1$ be arbitrary. We compute

$$\begin{aligned} S_m(T_\ell(f))(a, c) &= \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi ika} T_\ell(f) \left(ma, \frac{c+k}{m} \right) \\ &= \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi ika} \frac{1}{\ell} \sum_{j=0}^{\ell-1} f \left(\frac{ma+j}{\ell}, \ell \left(\frac{c+k}{m} \right) \right). \end{aligned}$$

(1) Now suppose $\ell = dm$. Then

$$S_m(T_{dm}(f))(a, c) = \frac{1}{dm^2} \sum_{k=0}^{m-1} e^{2\pi ika} \left(\sum_{j=0}^{dm-1} f \left(\frac{1}{d}a + \frac{j}{md}, d(c+k) \right) \right).$$

Now we apply twisted-periodicity to obtain

$$\begin{aligned} S_m(T_{dm}(f))(a, c) &= \frac{1}{dm^2} \sum_{j=0}^{dm-1} \left(\sum_{k=0}^{m-1} e^{2\pi ika} e^{-2\pi ikd \left(\frac{1}{d}a + \frac{j}{md} \right)} f \left(\frac{a}{d} + \frac{j}{md}, dc \right) \right) \\ &= \frac{1}{dm^2} \sum_{j=0}^{dm-1} f \left(\frac{a}{d} + \frac{j}{md}, dc \right) \left(\sum_{k=0}^{m-1} e^{-2\pi i \frac{jk}{m}} \right) \\ &= \frac{1}{dm} \sum_{n=0}^{d-1} f \left(\frac{a}{d} + \frac{n}{d}, dc \right) \\ &= \frac{1}{m} T_d(f)(a, c). \end{aligned}$$

Here we used on the second line the orthogonality relation that $\sum_{k=0}^{m-1} e^{-2\pi i \frac{jk}{m}} = 0$ unless m divides j and equals m if m divides j , and in the latter case, we write $m = jn$ in the next sum.

(2) When $d = 1$, we have $S_m(T_m(f))(a, c) = \frac{1}{m} T_1(f)(a, c)$. On the other hand,

$$\begin{aligned} T_m(S_m(f))(a, c) &= \frac{1}{m} \sum_{k=0}^{m-1} S_m(f) \left(\frac{a+k}{m}, mc \right) \\ &= \frac{1}{m} \sum_{k=0}^{m-1} \frac{1}{m} \sum_{j=0}^{m-1} e^{2\pi ij \left(\frac{a+k}{m} \right)} f \left(a+k, \frac{mc+j}{m} \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{m^2} \sum_{j=0}^{m-1} e^{2\pi i j(\frac{a}{m})} f\left(a, c + \frac{j}{m}\right) \left(\sum_{k=0}^{m-1} e^{2\pi i(\frac{jk}{m})}\right) \\
 &= \frac{1}{m} f(a, c) = \frac{1}{m} T_1(f)(a, c).
 \end{aligned}$$

Here we used the fact that f is invariant under translation by integers in a -variable on the third line, and the same orthogonality relation as in (1) on the fourth line. This establishes commutativity acting on equivalence classes of functions. Since $T_1(f) = f$ is the identity map, we have shown that T_m is both a left and right inverse to mS_m , whence T_m is invertible on $\mathcal{P}^*(\mathbb{R} \times \mathbb{R})$.

We then have $S_m = \frac{1}{m}(T_m)^{-1}$ on this domain. □

Lemma 6 *On the extended twisted-periodic function vector space $\mathcal{P}^*(\mathbb{R} \times \mathbb{R})$, the operators S_m and T_ℓ commute for all $\ell \geq 1, m \geq 1$.*

Proof We know that the operators T_ℓ mutually commute and that all the S_m mutually commute. By Lemma 5, we know that all T_ℓ are invertible operators on $\mathcal{P}(\mathbb{R} \times \mathbb{R})$, and that $S_m = \frac{1}{m}(T_m)^{-1}$ are also invertible operators. Thus

$$S_m \circ T_\ell = \frac{1}{m}(T_m)^{-1} \circ T_\ell = \frac{1}{m} T_\ell \circ (T_m)^{-1} = T_\ell \circ S_m$$

as required. □

Next recall that

$$T_m^\vee(f)(a, c) = \frac{1}{m} \left(\sum_{k=0}^{m-1} e^{2\pi i \left(\frac{(1-m)a+k}{m}\right)} f\left(\frac{a+k}{m}, 1+m(c-1)\right) \right) \tag{37}$$

and note that (36) can be rewritten

$$S_m^\vee(f)(a, c) = \frac{1}{m} \sum_{k=0}^{m-1} e^{-2\pi i(k+1)a} f\left(1+m(a-1), \frac{c+k}{m}\right).$$

Lemma 7 *On the extended twisted-periodic function space $\mathcal{P}^*(\mathbb{R} \times \mathbb{R})$, we have, for each $m \geq 1$, that*

$$T_m^\vee = T_m.$$

In addition,

$$S_m^\vee = S_m.$$

Proof (1) Under the assumption that f is a twisted-periodic function, we may rewrite (37) as the identity

$$T_m^\vee(f)(a, c) = e^{-2\pi ia} \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{a+k}{m}, m(c-1)\right). \tag{38}$$

We now show that

$$T_m^\vee \circ T_m(f)(a, c) = T_{m^2}(f)(a, c)$$

by calculating

$$\begin{aligned} T_m^\vee(T_m(f))(a, c) &= e^{-2\pi ia} \frac{1}{m} \sum_{k=0}^{m-1} T_m(f) \left(\frac{a+k}{m}, m(c-1) \right) \\ &= e^{-2\pi ia} \frac{1}{m} \sum_{k=0}^{m-1} \frac{1}{m} \left(\sum_{j=0}^{m-1} f \left(\frac{\frac{a+k}{m} + j}{m}, m(m(c-1)) \right) \right) \\ &= e^{-2\pi ia} \frac{1}{m^2} \sum_{\ell=0}^{m^2-1} f \left(\frac{a+\ell}{m^2}, m^2c - m^2 \right). \end{aligned}$$

We now apply the twisted-periodicity relations to obtain

$$\begin{aligned} T_m^\vee(T_m(f))(a, c) &= e^{-2\pi ia} \frac{1}{m^2} \sum_{\ell=0}^{m^2-1} e^{2\pi im^2 \left(\frac{a+\ell}{m^2} \right)} f \left(\frac{a+\ell}{m^2}, m^2c \right) \\ &= e^{-2\pi ia} \frac{1}{m^2} \sum_{\ell=0}^{m^2-1} e^{2\pi ia} f \left(\frac{a+\ell}{m^2}, m^2c \right) \\ &= T_{m^2}(f)(a, c), \end{aligned}$$

as asserted. Now we already know that $T_m \circ T_m = T_{m^2}$. Since the operator T_m is invertible on $\mathcal{P}^*(\mathbb{R} \times \mathbb{R})$, we conclude that $T_m^\vee = T_m$ on this domain.

We obtain $S_m^\vee = S_m$ by an analogous calculation; details are omitted.

4.4 L^p -spaces: Proof of Theorem 1

Analogous results on the two-variable Hecke operators T_m apply in the Banach spaces $L^p(\square, da dc)$ for $p \geq 1$. To study them, we note that functions on the extended function space $\mathcal{P}^*(\mathbb{R} \times \mathbb{R})$ are determined by their values in the open unit square \square° . We let

$$\mathcal{D}_p := \mathcal{P}^*(\mathbb{R} \times \mathbb{R})|_{\square^\circ} \cap L^p(\square, da dc).$$

This domain is dense in $L^p(\square, dadc)$ because it contains $C^0(\square)$, viewed as those continuous functions on \square° that continuously extend to \square , which is known to be dense ([36, Theorem 3.14]).

Lemma 8 *For all functions $F(a, c)$ in the domain $\mathcal{D}_p \subset L^p(\square, da dc)$, the L^p -norms of $T_m(F)$ and $S_m(F)$ restricted to \square° , for each $m \geq 1$, satisfy*

$$\|T_m(F)\|_p \leq m\|F\|_p \quad \text{and} \quad \|S_m(F)\|_p \leq m\|F\|_p$$

for all $p \geq 1$.

Proof The L^p -norm for $p < \infty$ is

$$\|F\|_p := \left(\int_0^1 \int_0^1 |F(a, c)|^p da dc \right)^{\frac{1}{p}},$$

and the norm $\|F\|_\infty$ for $p = \infty$ is the largest value among the suprema of $|F(a, c)|$ on the regions in \square° where it is continuous. For $p \geq 1$, this norm $\|\cdot\|_p$ satisfies the triangle inequality (by Minkowski’s inequality).

Write

$$(T_m F)(a, c) = \sum_{j=0}^{m-1} (T_m F)_j(a, c), \tag{39}$$

with

$$(T_m F)_j(a, c) := \begin{cases} T_m(F)(a, c) & \text{if } \frac{j}{m} \leq c < \frac{j+1}{m}, \\ 0 & \text{otherwise.} \end{cases}$$

Using twisted-periodicity, we find for $\frac{j}{m} \leq c < \frac{j+1}{m}$ that

$$(T_m F)_j(a, c) = \frac{1}{m} \sum_{k=0}^{m-1} F\left(\frac{a+k}{m}, mc-j\right) e^{-2\pi i j \left(\frac{a+k}{m}\right)}.$$

Now, for $1 \leq p < \infty$, letting

$$\tilde{F}\left(\frac{a+k}{m}, mc-j\right) := \begin{cases} F\left(\frac{a+k}{m}, mc-j\right) & \text{if } \frac{j}{m} \leq c < \frac{j+1}{m}, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} \|\tilde{F}\left(\frac{a+k}{m}, mc-j\right) e^{-2\pi i j \left(\frac{a+k}{m}\right)}\|_p^p &= \int_{\frac{j}{m}}^{\frac{j+1}{m}} \left(\int_0^1 \left| F\left(\frac{a+k}{m}, mc-j\right) \right|^p da \right) dc \\ &= \int_0^1 \int_{\frac{k}{m}}^{\frac{k+1}{m}} |F(\tilde{a}, \tilde{c})|^p d\tilde{a} d\tilde{c} \\ &\leq \int_0^1 \int_0^1 |F(\tilde{a}, \tilde{c})|^p d\tilde{a} d\tilde{c} = \|F\|_p^p. \end{aligned}$$

For $p = \infty$, a similar argument shows

$$\|\tilde{F}\left(\frac{a+k}{m}, mc-j\right) e^{-2\pi i j \left(\frac{a+k}{m}\right)}\|_\infty \leq \|F\|_\infty.$$

The L^p -triangle inequality then gives $\|(T_m F)_j\|_p \leq \frac{1}{m}(m\|F\|_p) = \|F\|_p$. Now (39) yields

$$\|T_m(F)\|_p \leq \sum_{j=0}^{m-1} \|(T_m F)_j\|_p \leq m\|F\|_p$$

for all $p \geq 1$, as required.

Interchanging the roles of a and c gives the same norm bound for

$$S_m(F)(a, c) = \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i k a} F\left(ma, \frac{c+k}{m}\right).$$

Proof of Theorem 1 (1) Earlier lemmas show the four families of operators $T_m, S_m, T_m^\vee, S_m^\vee$ preserve the extended twisted-periodic space $\mathcal{P}^*(\mathbb{R} \times \mathbb{R})$. The norm bound in Lemma 8 implies that they also preserve each domain \mathcal{D}_p , for $1 \leq p \leq \infty$. Lemmas 5, 6 and 7 together show that on each \mathcal{D}_p there are really only two distinct families of operators, which are T_m and S_m , and that $S_m = \frac{1}{m}(T_m)^{-1}$ holds there. It follows that the four families of operators $T_m, S_m, T_m^\vee, S_m^\vee$ induce well-defined bounded linear actions on $L^p(\square, da dc)$ for $1 \leq p \leq \infty$. Next for $F, G \in \mathcal{D}_p$ we have $S_m = \frac{1}{m}T_m^{-1}$ and Lemma 8 gives

$$\|F - G\|_p \leq m\|S_m(T_m(F)) - S_m(T_m(G))\|_p \leq m^2\|T_m(F) - T_m(G)\|_p.$$

Since \mathcal{D}_p is dense, taking limits gives the same norm bounds for $F, G \in L^p(\square, da dc)$. This lower bound inequality implies that T_m is invertible and that the relation $S_m = \frac{1}{m}T_m^{-1}$ continues to hold on the extended operators on $L^p(\square, da dc)$.

The argument above also establishes the following upper and lower bounds for the norm of $T_m(F)$:

$$\frac{1}{m^2} \|F\|_p \leq \|T_m(F)\|_p \leq m \|F\|_p. \tag{40}$$

We improve this bound for the case $p = 2$ below.

(2) The pairwise commutativity of members of all four sets of these operators on $L^p(\square, da dc)$ extends by continuity from their pairwise commutativity on the domain \mathcal{D}_p , which they preserve by (1), which itself follows from the commutativity result established in Lemma 6.

(3) On $L^2(\square, da dc)$, it suffices to show that the adjoint Hecke operator $(T_m)^* = S_m$, since applying $*$ then gives the other relation, whence \mathcal{A}_0^2 is a \star -algebra. We need only check it holds on the dense subspace \mathcal{D}_2 of $L^2(\square, da dc)$. We have

$$\begin{aligned} \langle T_m(f)(a, c), g(a, c) \rangle &= \frac{1}{m} \sum_{k=0}^{m-1} \int_0^1 \int_0^1 f\left(\frac{a+k}{m}, mc\right) \overline{g(a, c)} da dc \\ &= \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} \int_{\frac{j}{m}}^{\frac{j+1}{m}} \int_0^1 f\left(\frac{a+k}{m}, mc-j\right) e^{-2\pi ij(\frac{a+k}{m})} \overline{g(a, c)} da dc. \end{aligned}$$

Now we make the variable change $\tilde{c} = mc - j$ with $0 \leq \tilde{c} < 1$ and $\tilde{a} = \frac{a+k}{m}$ with $\frac{k}{m} \leq \tilde{a} < \frac{k+1}{m}$. Since $a = m\tilde{a} - k$ and $da dc = d\tilde{a}d\tilde{c}$, we obtain

$$\langle T_m(f)(a, c), g(a, c) \rangle = \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} \int_0^1 \left(\int_{\frac{k}{m}}^{\frac{k+1}{m}} f(\tilde{a}, \tilde{c}) e^{-2\pi ij\tilde{a}} \overline{g\left(m\tilde{a} - k, \frac{\tilde{c} + j}{m}\right)} d\tilde{a} \right) d\tilde{c}.$$

On the other hand

$$\begin{aligned} \langle f(a, c), S_m(g)(a, c) \rangle &= \frac{1}{m} \sum_{j=0}^{m-1} \int_0^1 \int_0^1 f(\tilde{a}, \tilde{c}) e^{2\pi ij\tilde{a}} \overline{g\left(m\tilde{a}, \frac{\tilde{c} + j}{m}\right)} d\tilde{a} d\tilde{c} \\ &= \frac{1}{m} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} \int_0^1 \int_{\frac{k}{m}}^{\frac{k+1}{m}} f(\tilde{a}, \tilde{c}) e^{-2\pi ij\tilde{a}} \overline{g\left(m\tilde{a} - k, \frac{\tilde{c} + j}{m}\right)} d\tilde{a} d\tilde{c}. \end{aligned}$$

Term-by-term comparison of the formulas shows $(T_m)^* = S_m$.

Now that we know the adjoint $(T_m)^* = S_m$, we have

$$\begin{aligned} \|T_m(F)\|_2^2 &= \langle T_m(F), T_m(F) \rangle_2 = \langle (T_m)^* \circ T_m(F), F \rangle_2 \\ &= \langle S_m \circ T_m(F), F \rangle_2 = \left\langle \frac{1}{m} T_m^{-1} \circ T_m(F), F \right\rangle_2 \\ &= \frac{1}{m} \|F\|_2^2. \end{aligned} \tag{41}$$

This yields the norm identity $\|T_m(F)\|_2 = \frac{1}{\sqrt{m}} \|F\|_2$ valid for all functions F . It follows that $\sqrt{m}T_m$ is a Hilbert space isometry, and since it is surjective, it is a unitary operator. In similar fashion, $\sqrt{m}S_m = \frac{1}{\sqrt{m}}T_m^{-1}$ is a surjective isometry, hence a unitary operator.

5 Lerch Eigenspaces \mathcal{E}_s and their properties

We construct a two-dimensional space \mathcal{E}_s of simultaneous eigenfunctions for all the two-variable Hecke operators, built out of the Lerch zeta function. We make use of the functional equations for the Lerch zeta function.

5.1 Lerch zeta function: functional equations

The functional equations proved in part I [28, Theorem 2.1] take a particularly simple form when expressed in terms of the R-operator. Recall that for $0 < a, c < 1$ these functional equations involve the two functions

$$L^+(s, a, c) = \sum_{n=-\infty}^{\infty} e^{2\pi ina} |n + c|^{-s}$$

and

$$L^-(s, a, c) = \sum_{n=-\infty}^{\infty} (\text{sgn}(n + c)) e^{2\pi ina} |n + c|^{-s}.$$

Now let $\Gamma_{\mathbb{R}}^+(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{R}}^-(s) = \pi^{\frac{s+1}{2}} \Gamma(\frac{s+1}{2}) = \Gamma_{\mathbb{R}}^+(s + 1)$. The functional equations are

$$\Gamma_{\mathbb{R}}^+(s) L^+(s, a, c) = e^{-2\pi iac} \Gamma_{\mathbb{R}}^+(1 - s) L^+(1 - s, 1 - c, a)$$

and

$$\Gamma_{\mathbb{R}}^-(s) L^-(s, a, c) = i e^{-2\pi iac} \Gamma_{\mathbb{R}}^-(1 - s) L^-(1 - s, 1 - c, a).$$

Definition 3 The *Tate gamma functions* $\gamma^{\pm}(s)$ (also known as *Gelfand–Graev gamma functions*, cf. [20, Chap. 2, Sect. 2.5]) are given by

$$\gamma^+(s) := \frac{\Gamma_{\mathbb{R}}^+(s)}{\Gamma_{\mathbb{R}}^+(1 - s)} = \frac{\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})}{\pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2})}, \tag{42}$$

$$\gamma^-(s) := \frac{\Gamma_{\mathbb{R}}^-(s)}{\Gamma_{\mathbb{R}}^-(1 - s)} = \frac{\pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2})}{\pi^{-\frac{2-s}{2}} \Gamma(\frac{2-s}{2})}. \tag{43}$$

These functions satisfy the identities

$$\gamma^{\pm}(s) \gamma^{\pm}(1 - s) = 1 \quad \text{for } s \in \mathbb{C}. \tag{44}$$

The functions $\gamma^{\pm}(1 - s) = \gamma^{\pm}(s)^{-1}$ have a “scattering matrix” interpretation (see Burnol [13, Sec. 5]), in which (44) expresses unitarity of the “scattering matrix.” The functions $\gamma^{\pm}(s)$ have simple zeros at certain positive integers and simple poles at certain negative integers, whose location depends on the sign \pm .

Recall also that part I ([28, Theorem 2.2]) defined the *extended Lerch functions* $L_{*}^{\pm}(s, a, c)$ on the domain $(a, c) \in \mathbb{R} \times \mathbb{R}$; these satisfy the twisted-periodicity conditions

$$L_{*}^{\pm}(s, a + 1, c) = L_{*}^{\pm}(s, a, c), \tag{45}$$

$$L_{*}^{\pm}(s, a, c + 1) = e^{-2\pi ia} L_{*}^{\pm}(s, a, c). \tag{46}$$

These functions are entire functions of s except for $L^+(s, a, c)$ when a or c is an integer, which is meromorphic with its only singularities being simple poles at $s = 0$ or $s = 1$, or both.

The R-operator acts on the Lerch functions as

$$R(L^{\pm})(s, a, c) = e^{-2\pi iac} L^{\pm}(s, 1 - c, a). \tag{47}$$

The functional equations for the Lerch zeta function given in part I have the following formulation in terms of the R-operator.

Proposition 2 (Lerch Functional Equations) *The functions $L^\pm(s, a, c)$ satisfy for $0 < a, c < 1$ the functional equations*

$$\Gamma_{\mathbb{R}}^\pm(s)L^\pm(s, a, c) = w_\pm \Gamma_{\mathbb{R}}^\pm(1-s)e^{-2\pi iac}L^\pm(1-s, 1-c, a), \tag{48}$$

in which $w_+ = 1, w_- = i$. Equivalently,

$$L^\pm(s, a, c) = w_\pm \gamma^\pm(1-s)R(L^\pm)(1-s, a, c). \tag{49}$$

The same functional equations hold for the extended Lerch functions $L_*^\pm(s, a, c)$ valid for $(a, c) \in \mathbb{R} \times \mathbb{R}$.

Proof For $0 < a, c < 1$, this result is shown in part I [28, Theorem 2.1], where we have restated that result using the R-operator. The extension to the boundary cases then uses the definition of $\zeta_*(s, a, c)$ given in part I. It applies on the domain $(a, c) \in \mathbb{R} \times \mathbb{R}$, given in [28, Theorem 2.2], including integer values of a and c .

5.2 Lerch eigenspaces \mathcal{E}_s

We will view the complex variable $s \in \mathbb{C}$ as fixed and introduce the abbreviated notations

$$L_s^\pm(a, c) := L^\pm(s, a, c) \tag{50}$$

$$R_s^\pm(a, c) := e^{-2\pi iac}L^\pm(1-s, 1-c, a). \tag{51}$$

The definition of the R-operator now gives

$$R_s^\pm(a, c) = R(L_{1-s}^\pm)(a, c). \tag{52}$$

The functional equations (49) given in Proposition 2 above can be rewritten

$$L_s^\pm(a, c) = w_\pm \gamma^\pm(1-s)R_s^\pm(a, c). \tag{53}$$

With this convention, the s -variable in $R_s^\pm(a, c)$ corresponds to $1-s$ in the associated Lerch function in (52), and this leads to $L_s^\pm(a, c)$ and $R_s^\pm(a, c)$ being linearly related in (53).

Definition 4 For fixed $s \in \mathbb{C}$, the *Lerch eigenspace* \mathcal{E}_s is the vector space over \mathbb{C} spanned by the four functions

$$\mathcal{E}_s := \langle L_s^+(a, c), L_s^-(a, c), R_s^+(a, c), R_s^-(a, c) \rangle \tag{54}$$

for $(a, c) \in \square^\circ = \{(a, c) : 0 < a, c < 1\}$.

The vector space \mathcal{E}_s is at most two-dimensional, since, as noted above, $L_s^+(a, c)$ and $R_s^+(a, c)$ (resp. $L_s^-(a, c)$ and $R_s^-(a, c)$) are linearly dependent, with dependency relations given by the functional equation (53).

Lemma 9 For each $s \in \mathbb{C}$, the *Lerch eigenspace* \mathcal{E}_s is a two-dimensional space all of whose members are real-analytic in (a, c) on the domain \square° .

Proof To see that \mathcal{E}_s is exactly two-dimensional for all $s \in \mathbb{C}$, note for fixed $\Re(s) > 0$ one has $\mathcal{E}_s = \langle L_s^\pm(a, c) \rangle$, with both functions of (a, c) being linearly independent. For fixed $\Re(s) < 1$, one has $\mathcal{E}_s = \langle R_s^\pm(a, c) \rangle$, with both functions of (a, c) being linearly independent. □

The reason for introducing four functions in (54) is that at integer values of s at least one of the four functions $L_s^\pm(a, c), R_s^\pm(a, c)$ is identically zero, corresponding to poles in the term $\gamma^\pm(1-s)$ in (49).

One can easily derive two alternate expressions for Lerch eigenspaces, valid for all $s \in \mathbb{C}$, which remove the effect of the gamma factors. They are

$$\mathcal{E}_s = \left\langle L^+(s, a, c), L^-(s, a, c), e^{-2\pi iac} L^+(1 - s, 1 - c, a), e^{-2\pi iac} L^-(1 - s, 1 - c, a) \right\rangle \tag{55}$$

and

$$\mathcal{E}_s = \left\langle \zeta_*(s, a, c), R(\zeta_*)(1 - s, a, c), R^2(\zeta_*)(s, a, c), R^3(\zeta_*)(1 - s, a, c) \right\rangle. \tag{56}$$

It is easy to check, using the functional equation, that each of the functions on the right side is in \mathcal{E}_s . It remains to verify that at each integer s one has two linearly independent elements in the (a, c) -variables.

5.3 Operator properties of Lerch eigenspaces \mathcal{E}_s

We show that the functions in the Lerch eigenspace are simultaneous eigenfunctions of the two-variable Hecke operators, the Lerch differential operator D_L and the involution operator $J = R^2$. We also show the vector spaces \mathcal{E}_s are permuted by related operators.

Theorem 5 (Operators on Lerch eigenspaces) *For each $s \in \mathbb{C}$, the functions in the two-dimensional vector space \mathcal{E}_s have the following properties.*

- (i) (Lerch differential operator eigenfunctions) *Each $f \in \mathcal{E}_s$ is an eigenfunction of the Lerch differential operator $D_L = \frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + c \frac{\partial}{\partial c}$ with eigenvalue $-s$, i.e.,*

$$D_L f(s, a, c) = -s f(s, a, c) \tag{57}$$

holds at all $(a, c) \in \mathbb{R} \times \mathbb{R}$, with both a and c nonintegers.

- (ii) (Simultaneous Hecke operator eigenfunctions) *Each $f \in \mathcal{E}_s$ is a simultaneous eigenfunction of all two-variable Hecke operators $\{T_m : m \geq 1\}$ with eigenvalue m^{-s} , in the sense that*

$$T_m f = m^{-s} f \tag{58}$$

holds on the domain $(\mathbb{R} \setminus \frac{1}{m}\mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z})$.

- (iii) (J-operator eigenfunctions) *The space \mathcal{E}_s admits the involution $Jf(a, c) = e^{-2\pi ia} f(1 - a, 1 - c)$, under which it decomposes into one-dimensional eigenspaces $\mathcal{E}_s = \mathcal{E}_s^+ \oplus \mathcal{E}_s^-$ with eigenvalues ± 1 , namely $\mathcal{E}_s^\pm = \langle F_s^\pm \rangle$ and*

$$J(F_s^\pm) = \pm F_s^\pm. \tag{59}$$

- (iv) (R-operator action) *The R-operator acts by $R(\mathcal{E}_s) = \mathcal{E}_{1-s}$ with*

$$R(L_s^\pm) = w_\pm^{-1} \gamma^\pm (1 - s) L_{1-s}^\pm, \tag{60}$$

where $w_+ = 1$ and $w_- = i$.

This result yields the following consequence.

Corollary 1 (Invariance of \mathcal{E}_s under Hecke operator families) *For each $s \in \mathbb{C}$, the Lerch eigenspace \mathcal{E}_s is invariant under all four families of real-analytic two-variable Hecke operators $\{T_m : m \geq 1\}$, $\{T_m^\vee : m \geq 1\}$, $\{S_m : m \geq 1\}$, and $\{S_m^\vee : m \geq 1\}$.*

Proof These four sets of operators are $\{R^j T_m R^{-j}\}$ for $j = 0, 1, 2, 3$. The case $j = 0$ is covered by (ii). By (iv) the effect of R is to map the four generating functions for \mathcal{E}_s to a

permutation of them spanning \mathcal{E}_{1-s} . Since the R operator is applied an even number of times for $j = 1, 2, 3$, the final image is always \mathcal{E}_s .

To prove Theorem 5, we first determine the action of D_L^+ and D_L^- on the eigenspace \mathcal{E}_s .

Lemma 10 (Raising and lowering actions)

(i) For each $s \in \mathbb{C}$, the operator $D_L^- = \frac{1}{2\pi i} \frac{\partial}{\partial a} + c$ has

$$(D_L^- L_s^\pm)(s, a, c) = L_{s-1}^\mp(s, a, c), \tag{61}$$

so that it takes $D_L^-(\mathcal{E}_s) = \mathcal{E}_{s-1}$.

(ii) For each $s \in \mathbb{C}$, the operator $D_L^+ = \frac{\partial}{\partial c}$ has

$$(D_L^+ L_s^\pm)(s, a, c) = -sL_{s+1}^\mp(s, a, c), \tag{62}$$

so that it takes $D_L^+(\mathcal{E}_s) = \mathcal{E}_{s+1}$.

Proof We establish (i) and (ii) for $\Re(s) > 1$ using the basis $L_s^\pm(a, c)$ for \mathcal{E}_s and the Dirichlet series representations

$$\zeta(s, a, c) = \sum_{n=0}^{\infty} e^{2\pi i na} (n + c)^{-s} \tag{63}$$

and

$$e^{-2\pi ia} \zeta(s, 1 - a, 1 - c) = \sum_{n=1}^{\infty} e^{-2\pi ina} (n - c)^{-s}. \tag{64}$$

Calculations can be done term-by-term on these Dirichlet series to verify (61) and (62). For real $0 < a < 1$ and $0 < c < 1$, the Dirichlet series (63) converges conditionally for $\Re(s) > 0$ and differentiating term-by-term can still be justified. (Alternatively, we can analytically continue in the s -variable.)

For $\Re(s) < 0$ (resp. $\Re(s) < 1$), we apply similar reasoning to the basis $R_s^\pm(a, c)$ for \mathcal{E}_s , which is possible since these functions have absolutely (resp. conditionally) convergent Dirichlet series expansion in this range. \square

Proof (i) Since $D_L = D_L^- D_L^+$, by Lemma 10 the relation

$$D_L L_s^\pm = -sL_s^\pm \tag{65}$$

follows for all $s \in \mathbb{C}$. Since $\mathcal{E}_s = \langle L_s^+(a, c), L_s^-(a, c) \rangle$ for $\Re(s) > 0$, this proves (i) for $\Re(s) > 0$. In the case $\Re(s) < 1$, we use the alternate basis

$$\mathcal{E}_s = \left\langle e^{-2\pi iac} L_{1-s}^+(1 - c, a), e^{-2\pi iac} L_{1-s}^-(1 - c, a) \right\rangle,$$

see (55).

Since

$$e^{-2\pi iac} L_{1-s}^\pm(1 - c, a) = R(L_{1-s}^\pm(a, c)),$$

we may apply Lemma 3 (iii) and (65) to get

$$\begin{aligned} D_L(R(L_{1-s}^\pm(a, c))) &= -RD_L(L_{1-s}^\pm(a, c)) - RL_{1-s}^\pm(a, c) \\ &= -R(-(1 - s)L_{1-s}^\pm(a, c)) - RL_{1-s}^\pm(a, c) \\ &= -sR(L_{1-s}^\pm(a, c)). \end{aligned}$$

Thus (i) follows in this case.

(ii) It suffices to verify that $T_m f = m^{-s} f$ on a basis of \mathcal{E}_s . For $\Re(s) > 0$, we use the basis $\mathcal{E}_s = \langle \zeta(s, a, c), e^{-2\pi i a} \zeta(s, 1 - a, 1 - c) \rangle$. Using the Dirichlet series expansion (63), we have, formally,

$$\begin{aligned} T_m \zeta(s, a, c) &= \frac{1}{m} \sum_{k=0}^{m-1} \zeta\left(s, \frac{a+k}{m}, mc\right) \\ &= \frac{1}{m} \sum_{k=0}^{m-1} \sum_{n=0}^{\infty} e^{2\pi i n \left(\frac{a+k}{m}\right)} (n+mc)^{-s} \\ &= \sum_{n=0}^{\infty} e^{2\pi i \frac{na}{m}} \left(\frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i \frac{kn}{m}} \right) (n+mc)^{-s}. \end{aligned}$$

The inner sum is 0 unless $n \equiv 0 \pmod{m}$, in which case it is 1, so that, setting $n = ml$, we obtain

$$T_m \zeta(s, a, c) = \sum_{l=0}^{\infty} e^{2\pi i la} (m(l+c))^{-s} = m^{-s} \zeta(s, a, c).$$

These operations are easily justified when $\Re(s) > 1$ and the sum converges absolutely. They also hold for $0 < \Re(s) \leq 1$ when the sums converge conditionally, by using finite sums

$$\sum_{n=0}^{Nm} e^{2\pi i n \left(\frac{a+k}{m}\right)} (n+mc)^{-s}$$

and then letting $N \rightarrow \infty$. A similar argument shows that

$$T_m(e^{-2\pi i a} \zeta(s, 1 - a, 1 - c)) = m^{-s} e^{-2\pi i a} \zeta(s, 1 - a, 1 - c)$$

when $\Re(s) > 0$, and this completes the case $\Re(s) > 0$. For the case $\Re(s) < 1$ we work with a different basis of \mathcal{E}_s :

$$\mathcal{E}_s = \left\langle e^{-2\pi i a c} \zeta(1 - s, 1 - c, a), e^{-2\pi i a c + 2\pi i c} \zeta(1 - s, c, 1 - a) \right\rangle.$$

The proof is actually simpler. We shall show details for $e^{-2\pi i a c} \zeta(1 - s, 1 - c, a) := F(a, c)$, the other basis function being similar. First assume that $\Re(s) < 0$ so that $\Re(1 - s) > 1$ and $F(a, c)$ is given by an absolutely convergent series

$$F(a, c) = e^{-2\pi i a c} \sum_{n=0}^{\infty} e^{2\pi i n(1-c)} (n+a)^{-(1-s)}.$$

Then

$$\begin{aligned} T_m F(a, c) &= \frac{1}{m} \sum_{k=0}^{m-1} e^{-2\pi i(a+k)c} \sum_{n=0}^{\infty} e^{2\pi i n(1-mc)} \left(n + \frac{a+k}{m} \right)^{-(1-s)} \\ &= \frac{1}{m} m^{1-s} e^{-2\pi i a c} \sum_{n=0}^{\infty} \sum_{k=0}^{m-1} (mn+k+a)^{-(1-s)} e^{-2\pi i c(mn+k)} \\ &= m^{-s} F(a, c), \end{aligned}$$

as desired. For $0 \leq \Re(s) < 1$, the series for $F(a, c)$ converges conditionally and we use the partial sum $\sum_{n=0}^N$ and let $N \rightarrow \infty$ to approach $F(a, c)$ as in the case $0 < \Re(s) \leq 1$. This proves (ii)

(iii) Using $Jf(a, c) = e^{-2\pi ia}f(1 - a, 1 - c)$ and the definition (50) of L_s^\pm , it is easy to see that

$$J(L_s^\pm) = \pm L_s^\pm \tag{66}$$

holds for all $s \in \mathbb{C}$. This shows $J : \mathcal{E}_s \rightarrow \mathcal{E}_s$ for all $s \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, where $\mathcal{E}_s = \langle F_s^+, F_s^- \rangle$, and gives a splitting $\mathcal{E}_s = \mathcal{E}_s^+ \oplus \mathcal{E}_s^-$. For the remaining values, we use the basis $R_s^\pm(a, c)$ for \mathcal{E}_s , and a straightforward computation shows that

$$J(R_s^\pm) = \pm R_s^\pm \tag{67}$$

holds for all $s \in \mathbb{C}$. This shows that $J : \mathcal{E}_s \rightarrow \mathcal{E}_s$ for $s \in \mathbb{C} \setminus \{1, 2, 3, \dots\}$ where $\mathcal{E}_s = \langle R_s^+, R_s^- \rangle$, and it gives a splitting $\mathcal{E}_s = \mathcal{E}_s^+ \oplus \mathcal{E}_s^-$. The functions L_s^+ and R_s^+ (resp. L_s^- and R_s^-) are proportional for each $s \in \mathbb{C}$ by the functional equation.

(iv) The relation

$$w_\pm R(L_s^\pm)(a, c) = \gamma^\pm(s)L_{1-s}^\pm(a, c) \tag{68}$$

with $w_+ = 1$ and $w_- = i$ follows from the symmetrized functional equation (49) by replacing s with $1 - s$. Since $R^4 = I$, the image of the vector space \mathcal{E}_s is necessarily a vector space $R(\mathcal{E}_s)$ of the same dimension. We know that $\dim(\mathcal{E}_s) = 2$ by Lemma 9.

If $s \in \mathbb{C} \setminus \mathbb{Z}$, then the coefficient on the right side of (68) is in $\mathbb{C} \setminus \{0\}$ for both choices of sign \pm , which exhibits an explicit isomorphism $R : \mathcal{E}_s \rightarrow \mathcal{E}_{1-s}$. If $s = s_0 \in \mathbb{Z}$, then exactly one of $\Gamma_{\mathbb{R}}^\pm(s)$ and $\Gamma_{\mathbb{R}}^\pm(1 - s)$ has a pole at s_0 , and (68) is not well defined. What happens is that one of $L_s^\pm(a, c)$ or $L_{1-s}^\pm(a, c)$ is identically zero. This can be seen by studying the functional equation (68) as s varies, approaching the value s_0 . One side of the equation is defined and finite at s_0 ; the other side must, for each fixed $(a, c) \in (0, 1) \times (0, 1)$, have the corresponding $F_s^\pm(a, c) \rightarrow 0$ as $s \rightarrow s_0$, since it is an entire function of s . However, we can still see that $R(\mathcal{E}_{s_0}) = \mathcal{E}_{1-s_0}$, by noting that $R(\mathcal{E}_s) \subseteq \mathcal{E}_{1-s}$ for $s \notin \mathbb{Z}$, and letting $s \rightarrow s_0$ we obtain $R(\mathcal{E}_{s_0}) \subseteq \mathcal{E}_{1-s_0}$, by analytic continuation in s . The image is necessarily two-dimensional, since R is invertible, so $R(\mathcal{E}_{s_0}) = \mathcal{E}_{1-s_0}$ holds for $s = s_0$. \square

5.4 Analytic properties of Lerch eigenspaces \mathcal{E}_s

We establish the following analytic properties of members of \mathcal{E}_s , which we deduce from results in part I.

Theorem 6 (Analytic properties of Lerch eigenspaces) *For fixed $s \in \mathbb{C}$, the functions in the Lerch eigenspace \mathcal{E}_s are real-analytic functions of (a, c) on $(\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z})$, which may be discontinuous at values $a, c \in \mathbb{Z}$. They have the following properties.*

(i) (Twisted-Periodicity Property) *All functions $F(a, c)$ in \mathcal{E}_s satisfy the twisted-periodic functional equations*

$$F(a + 1, c) = F(a, c), \tag{69}$$

$$F(a, c + 1) = e^{-2\pi ia}F(a, c). \tag{70}$$

(ii) (Integrability Properties)

(a) *If $\Re(s) > 0$, then for each noninteger c all functions in \mathcal{E}_s have $f_c(a) := F(a, c) \in L^1[(0, 1), da]$, and all their Fourier coefficients*

$$f_n(c) := \int_0^1 F(a, c)e^{-2\pi ina} da, \quad n \in \mathbb{Z}, \tag{71}$$

are continuous functions of c on $0 < c < 1$.

(b) If $\Re(s) < 1$, then for each noninteger a all functions in \mathcal{E}_s have $g_a(c) := e^{2\pi iac} F(a, c) \in L^1[(0, 1), dc]$, and all Fourier coefficients

$$g_n(a) := \int_0^1 e^{2\pi iac} F(a, c) e^{-2\pi inc} dc, \quad n \in \mathbb{Z}, \tag{72}$$

are continuous functions of a on $0 < a < 1$.

(c) If $0 < \Re(s) < 1$ then all functions in \mathcal{E}_s belong to $L^1[\square, dadc]$.

Proof These properties will be deduced from results in part I.

- (i) Theorem 2.2 of part I [28] established the twisted-periodicity functional equations for $\zeta_*(s, a, c)$. It follows by repeated applications of R that these functional equations also hold for $e^{-2\pi iac} \zeta(1-s, 1-c, a)$, $e^{-2\pi ia} \zeta(s, 1-a, 1-c)$, $e^{-2\pi ia(c+1)} \zeta(1-s, c, 1-a)$. These four functions span the two-dimensional vector space \mathcal{E}_s for every $s \in \mathbb{C}$.
- (ii) Part I [28, Theorem 6.1] shows for $s \in \mathbb{C} \setminus \mathbb{Z}$, that subtracting off suitable members of the four basis functions

$$c^{-s}, e^{-2\pi ia}(1-c)^{-s}, e^{-2\pi i(1-a)c}(1-a)^{s-1}, e^{-2\pi iac} a^{s-1} \tag{73}$$

from the two functions $L^\pm(s, a, c)$ yields functions $\tilde{L}^\pm(s, a, c)$ that are continuous on the closed unit square $\square = [0, 1] \times [0, 1]$, and which therefore belong to $L^2[\square, dadc]$. Since

$$\zeta_*(s, a, c) = L^+(s, a, c) + L^-(s, a, c),$$

it also has a continuous extension to the closed unit square after subtracting off suitable multiples of these four functions. In fact, only three of the four basis functions are needed in the subtraction, for $\zeta_*(s, a, c)$ the function $e^{-2\pi ia}(1-c)^{-s}$ is omitted, see [28, Theorems 5.1 and 5.2]. At the integer values of s excluded, some of the terms subtracted off have poles.

We now consider the effect of the singularities of these basis functions on the four sides of the unit square on determining absolute integrability of $\zeta_*(s, a, c)$ on horizontal and vertical lines in the unit square. In the a -direction, for all $s \in \mathbb{C}$ the two functions c^{-s} and $e^{-2\pi ia}(1-c)^{-s}$ each lie in $L^1[(0, 1), da]$ for each fixed value $0 < c < 1$, while for $\Re(s) > 0$ the two functions $e^{-2\pi iac}(1-a)^{s-1}$ and $e^{-2\pi iac+2\pi ic} a^{s-1}$ lie in $L^1[(0, 1), da]$ for each fixed value $0 < c < 1$. In the c -direction, for $\Re(s) < 1$ the two functions c^{-s} , $e^{-2\pi ia}(1-c)^{-s}$ lie in $L^1[(0, 1), dc]$ for each fixed value $0 < a < 1$, while for all $s \in \mathbb{C}$ the other two functions lie in $L^1[(0, 1), dc]$ for each fixed value $0 < a < 1$. These properties are inherited by $\zeta_*(s, a, c)$ which establishes the L^1 -membership part of (ii-a) and (ii-b) for $\zeta_*(s, a, c)$ when $s \in \mathbb{C} \setminus \mathbb{Z}$. Now we apply the R operator, interchanging s and $1-s$ as necessary, and deduce that (ii-a) and (ii-b) also hold for the other three functions in (56). Since these functions span \mathcal{E}_s , the L^1 -membership properties hold for all functions in \mathcal{E}_s . For the Fourier coefficient assertion, the formula (8) gives a convergent Fourier series in the a -variable for $\zeta_*(s, a, c)$ valid for $\Re(s) > 0$, and the continuity of the Fourier coefficients $f_n(c) = (n+c)^{-s}$ for $n \geq 0, f_n(c) = 0$ for $n < 0$ is manifest. Similarly, we obtain continuity of the Fourier coefficients in the a -variable of $e^{-2\pi ia} \zeta_*(s, 1-a, 1-c)$ when $\Re(s) > 0$. These two functions span \mathcal{E}_s for $\Re(s) > 0$, which establishes (ii-a) for $s \in \mathbb{C} \setminus \mathbb{Z}$. The Fourier coefficient assertion (ii-b) is obtained by similar calculations. Multiplication by $e^{2\pi iac}$ does not affect membership in $L^1[(0, 1), dc]$.

One finds for $F(a, c) = R(\zeta_*)(1 - s, a, c) \in \mathcal{E}_s$ that $e^{2\pi iac}F(a, c) = \zeta_*(1 - s, 1 - c, a)$ has Fourier coefficients in the c -variable $g_n(a) = |-n + a|^{s-1}$ for $n \geq 0$ and $g_n(a) = 0$ for $n < 0$, while $F(a, c) = R^3(\zeta_*)(1 - s, a, c) \in \mathcal{E}_s$ has $e^{2\pi ic}F(a, c) = e^{2\pi ic}\zeta_*(1 - s, c, 1 - a)$ and has Fourier coefficients in c -variable $g_n(a) = 0$ for $n \geq 0$ and $g_n(a) = |-n + a|^{s-1}$ for $n < 0$. These two functions span \mathcal{E}_s for $\Re(s) < 1$, which establishes (ii-b) for $s \in \mathbb{C} \setminus \mathbb{Z}$.

We next establish the remaining cases of properties (ii-a) and (ii-b), which are (ii-a) for integer $s \geq 1$ and (ii-b) for integer $s \leq -1$. For (ii-a), we directly use the Fourier series expansion (8). After removing the term c^{-s} , which is clearly in $L^1[(0, 1), da]$, the remaining Fourier series for fixed $0 < c < 1$ is absolutely integrable at all integers $s = n \geq 2$, and its Fourier coefficients are continuous in c by inspection. A similar property holds for $\zeta_*(s, 1 - a, 1 - c)$, after removing the term $e^{-2\pi ia}(1 - c)^{-s}$. Since for $\Re(s) > 0$ these functions span \mathcal{E}_s , property (ii-a) holds in these cases. For the remaining case $s = 1$, Rohrlich showed (see Milnor [32, Lemma 4]) that both the functions $\zeta_*(1, a, c) - c^{-1} = \sum_{n=1}^{\infty} \frac{e^{2\pi ina}}{n+c}$ and $\zeta_*(1, 1 - a, 1 - c) - \frac{e^{2\pi ia}}{(1-c)}$ are in $L^1[(0, 1), da]$ with the given Fourier coefficients, completing this case. The corresponding property (ii-b) for $s = n \leq -1$ is established similarly.

Property (ii-c) is shown in part I [28, Theorem 2.4]. □

6 Hecke Eigenfunction characterization of Lerch Eigenspaces \mathcal{E}_s

In this section, we characterize the Lerch eigenspace as being the complete set of simultaneous eigenfunctions of the family of two-variable Hecke operators $\{T_m : m \geq 1\}$ that satisfy some auxiliary integrability and continuity conditions on the function. This can be viewed as a generalizing Milnor’s characterization of space of Hurwitz zeta functions, which we first explain in §6.1. Our characterization theorem is given in §6.2. We note that this characterization theorem does not impose any eigenfunction condition with respect to the differential operator D_L .

6.1 Milnor’s theorem for Kubert functions

A Kubert operator T_m is an operator that acts formally as

$$T_m(f)(x) := \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right). \tag{74}$$

These operators were studied by Kubert and Lang [24,25] and Kubert [23], acting on a group, for example \mathbb{R}/\mathbb{Z} . In 1983, Milnor [32] characterized simultaneous eigenfunction solutions to the Kubert operator, acting on continuous functions on the open interval $(0, 1)$.

Milnor studied the family of operators $T_m : C^0((0, 1)) \rightarrow C^0((0, 1))$ given by (74). These operators form a commuting family of operators on $C^0((0, 1))$.

Theorem 7 (Milnor) *Let \mathcal{K}_s denote the set of continuous functions $f : (0, 1) \rightarrow \mathbb{C}$ which satisfy*

$$T_m f(x) = m^{-s} f(x) \quad \text{for all } x \in (0, 1), \tag{75}$$

for each $m \geq 1$. Then \mathcal{K}_s is a two-dimensional complex vector space and consists of real-analytic functions. Furthermore, \mathcal{K}_s is an invariant subspace for the involution

$$J_0 f(x) := f(1 - x)$$

and decomposes into one-dimensional eigenspaces $\mathcal{K}_s = \mathcal{K}_s^+ \oplus \mathcal{K}_s^-$ which are spanned by an even eigenfunction $f_s^+(x)$ and an odd eigenfunction $f_s^-(x)$, respectively, which satisfy

$$J_0 f_s^\pm(x) = \pm f_s^\pm(x). \tag{76}$$

Proof This is proved in [32, Theorem 1].

Milnor gave an explicit basis for \mathcal{K}_s , which for $s \neq 0, -1, -2, \dots$ is given in terms of the (analytic continuation in s of the) Hurwitz zeta function

$$\zeta_s(x) := \zeta(s, 0, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \tag{77}$$

namely

$$\mathcal{K}_s = \langle \zeta_{1-s}(x), \zeta_{1-s}(1-x) \rangle. \tag{78}$$

Properties (ii) and (iii) of Theorem 5 are analogous to those in Theorem 7, in which the variable x in the Kubert operator (74) is identified with the variable a , and the second variable c is set to 0.

Milnor observes that $\frac{\partial}{\partial x}$ maps \mathcal{K}_s to \mathcal{K}_{s-1} , acting as a “lowering operator.” Because the individual operators inside the sum on the right side of (74) are contracting, this “lowering operator” suffices in his proof.

In §3 we observed, in the two-variable context, that $\frac{1}{2\pi i} \frac{\partial}{\partial a} + c : \mathcal{E}_s \rightarrow \mathcal{E}_{s-1}$ is a “lowering” operator, while $\frac{\partial}{\partial c} : \mathcal{E}_s \rightarrow \mathcal{E}_{s+1}$ is a “raising” operator. Property (i) of Theorem 5 is derived using these properties. Milnor’s theorem *formally* corresponds to setting $a = x$ and $c = 0$ in Theorem 5, except that $c = 0$ falls outside the domain of definition of the functions we consider.

6.2 Characterization of Lerch eigenspaces \mathcal{E}_s

Milnor’s proof of Theorem 7 used in an essential way the property that for “Kubert operators” T_m all terms on the right side of (74) are contracting operators on the domain $x \in (0, 1)$.

In contrast, the two-variable Hecke operators (6) are expanding in the c -direction. To deal with the expanding property, we impose extra analytic conditions on the function in the whole plane $\mathbb{R} \times \mathbb{R}$, in order to obtain a characterization of \mathcal{E}_s as being simultaneous eigenfunctions of two-variable Hecke operators.

Our main result shows that the twisted-periodicity and integrabilities properties of Theorem 6 yield such a characterization.

Theorem 8 (Lerch Eigenspace Characterization) *Let $s \in \mathbb{C}$. Suppose that $F(a, c) : (\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z}) \rightarrow \mathbb{C}$ is a continuous function that satisfies the following conditions.*

- (1) (Twisted-Periodicity Condition) *For $(a, c) \in (\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z})$,*

$$F(a + 1, c) = F(a, c), \tag{79}$$

$$F(a, c + 1) = e^{-2\pi i a} F(a, c). \tag{80}$$

- (2) (Integrability Condition) *At least one of the following two conditions (2-a) or (2-c) holds.*

(2-a) The s -variable has $\Re(s) > 0$. For $0 < c < 1$ each function $f_c(a) := F(a, c) \in L^1[(0, 1), da]$, and all the Fourier coefficients

$$f_n(c) := \int_0^1 f_c(a)e^{-2\pi ina} da = \int_0^1 F(a, c)e^{-2\pi ina} da, \quad n \in \mathbb{Z},$$

are continuous functions of c .

(2-c) The s -variable has $\Re(s) < 1$. For $0 < a < 1$ each function $g_a(c) := e^{2\pi iac}F(a, c) \in L^1[(0, 1), dc]$, and all the Fourier coefficients

$$g_n(a) := \int_0^1 g_a(c)e^{-2\pi inc} dc = \int_0^1 e^{2\pi iac}F(a, c)e^{-2\pi inc} dc, \quad n \in \mathbb{Z},$$

are continuous functions of a .

(3) (Hecke Eigenfunction Condition) For all $m \geq 1$,

$$T_m(F)(a, c) = m^{-s}F(a, c) \tag{81}$$

holds on the domain $\{(a, c) \in (\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \frac{1}{m}\mathbb{Z})\}$.

Then $F(a, c)$ is the restriction to noninteger (a, c) -values of a function in the Lerch eigenspace \mathcal{E}_s .

Remarks. (i) Theorem 6 shows that all functions in \mathcal{E}_s for $\Re(s) > 0$ satisfy conditions (1), (2-a) and (3) and all functions in \mathcal{E}_s for $\Re(s) < 1$ satisfy conditions (1), (2-c) and (3) above. Conditions (2-a), (2-c) between them cover all $s \in \mathbb{C}$, and they hold simultaneously inside the critical strip $0 < \Re(s) < 1$.

(ii) The function $F(a, c) := c^{-s}$ satisfies properties (2-a) and (2-c) and also the eigenvalue property (3). However, it fails to satisfy the twisted-periodicity property (1).

Proof We first treat the case when condition (2-a) holds, where $F(a, c)$ is absolutely integrable on horizontal lines in the unit square. The proof uses the Fourier series expansion of $F(a, c)$ with respect to the a -variable, which we write as

$$F(a, c) \sim \sum_{n=-\infty}^{\infty} f_n(c)e^{2\pi ina}, \tag{82}$$

where \sim means that no assertion is made about convergence of the Fourier series to $F(a, c)$.

Condition (2-a) shows that all the Fourier coefficients

$$f_n(c) := \int_0^1 F(a, c)e^{-2\pi ina} da \tag{83}$$

are continuous functions of c for $0 < c < 1$. The Fourier coefficients are also defined for $\ell < c < \ell + 1$ for all integer ℓ , using the twisted-periodicity property (80) in the c -variable, $F(a, c + \ell) = e^{-2\pi i\ell a}F(a, c)$, and they satisfy

$$f_n(c + \ell) = f_{n+\ell}(c) \quad \text{for all } \ell \in \mathbb{Z} \tag{84}$$

by uniqueness of the Fourier series expansion for L^1 -functions.

The action of the two-variable Hecke operator

$$T_m(F)(a, c) := \frac{1}{m} \sum_{k=0}^{m-1} F\left(\frac{a+k}{m}, mc\right)$$

is well-defined pointwise for $(a, c) \in (\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \frac{1}{m}\mathbb{Z})$. The resulting function is in $L^1[(0, m), da]$ for $c \in \mathbb{R} \setminus \frac{1}{m}\mathbb{Z}$ and is periodic of period 1 in the a -variable because

$$T_m(F)(a + 1, c) = T_m(F)(a, c) + \frac{1}{m} \left(F \left(\frac{a + m}{m}, mc \right) - F \left(\frac{a}{m}, mc \right) \right) = T_m(F)(a, c)$$

by (79). Its Fourier expansion on $L^2[(0, 1), da]$ is

$$\begin{aligned} T_m(F)(a, c) &= \frac{1}{m} \sum_{k=0}^{m-1} \sum_{n \in \mathbb{Z}} f_n(mc) e^{-2\pi i n (\frac{a+k}{m})} \\ &= \sum_{n \in \mathbb{Z}} \left(\frac{1}{m} \sum_{k=0}^{m-1} e^{-2\pi i n (\frac{a+k}{m})} \right) f_n(mc) \\ &= \sum_{\ell \in \mathbb{Z}} f_{m\ell}(mc) e^{-2\pi i \ell a}. \end{aligned} \tag{85}$$

Now suppose that $T_m(F)(a, c) = m^{-s}F(a, c)$ on the indicated domain. Comparison of (85) with the Fourier series expansion of $m^{-s}F(a, c)$ in (82) gives

$$f_{mn}(mc) = m^{-s}f_n(c) \quad \text{for } c \in \mathbb{R} \setminus \frac{1}{m}\mathbb{Z}. \tag{86}$$

To simplify later formulas, we set

$$\tilde{f}_n(c) := |n + c|^s f_n(c). \tag{87}$$

Then (84) and (87) yield, for all $l \in \mathbb{Z}$,

$$\tilde{f}_n(c + \ell) = f_n(c + \ell) |n + c + \ell|^s = f_{n+\ell}(c) |n + c + \ell|^s = \tilde{f}_{n+\ell}(c). \tag{88}$$

Furthermore, (86) gives

$$\begin{aligned} \tilde{f}_{mn}(mc) &= f_{mn}(mc) |mn + mc|^s \\ &= (m^{-s}f_n(c)) m^s |n + c|^s \\ &= \tilde{f}_n(c). \end{aligned} \tag{89}$$

We now determine all solutions to (88) and (89). Consider $n = 0$, and we obtain, for all $m \geq 1$,

$$\tilde{f}_0(mc) = \tilde{f}_0(c). \tag{90}$$

The right side of (90) is continuous for $c \in (0, 1)$, which implies that the left side makes sense as a continuous function for $mc \in (0, m)$. Since m is arbitrarily large, we conclude that $\tilde{f}_0(c)$ extends to a continuous function on $(0, \infty)$. Now (84) gives

$$f_0(c - 1) = f_{-1}(c) \quad \text{for } 0 < c < 1,$$

and condition (2-a) gives the continuity of $f_{-1}(c)$ on this interval, so it follows that $\tilde{f}_0(c)$ is continuous on $(-1, 0)$. As above (90) implies that $\tilde{f}_0(c)$ extends to a continuous function on $(-\infty, 0)$. Thus any possible discontinuity of $\tilde{f}_0(c)$ is at $c = 0$. Now writing $\tilde{c} = \frac{m_1}{m_2}c$ for positive integers m_1, m_2 and applying (90), we obtain for positive \tilde{c} that

$$\tilde{f}_0(\tilde{c}) = \tilde{f}_0(m_2\tilde{c}) = \tilde{f}_0(m_1c) = \tilde{f}_0(c)$$

and similarly for negative \tilde{c} . Thus $\tilde{f}_0(c) = \tilde{f}_0(rc)$ for all positive rational numbers r , which with the continuity results implies that $\tilde{f}_0(c)$ is constant on $(-\infty, 0)$ and on $(0, \infty)$, say $\tilde{f}_0(c) = A$ (resp. B) on $(0, \infty)$ (resp. $(-\infty, 0)$). Thus we obtain

$$f_0(c) = \begin{cases} A|c|^{-s} & \text{if } c > 0 \\ B|c|^{-s} & \text{if } c < 0. \end{cases}$$

Now (84) gives

$$f_n(c) = f_0(c + n) = \begin{cases} A|c + n|^{-s} & \text{if } c > -n, \\ B|c + n|^{-s} & \text{if } c < -n. \end{cases}$$

Thus the Fourier series of $F(a, c)$ agrees term-by-term with the Fourier series of

$$H(a, c) := \frac{1}{2}(A + B)L^+(s, a, c) + \frac{1}{2}(A - B)L^-(s, a, c). \tag{91}$$

Since we have $\Re(s) > 0$, this is in $L^1[(0, 1), da]$ for noninteger c . So by uniqueness of Fourier series and continuity we conclude that $F(a, c) = H(a, c) \in \mathcal{E}_s$ everywhere on $(\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z})$.

We now treat the case that condition (2-c) holds. The proof is similar in spirit. We set

$$G(a, c) := e^{2\pi iac}F(a, c) \text{ for } (a, c) \in (\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z}).$$

This function satisfies the modified twisted-periodicity conditions.

$$G(a + 1, c) = e^{2\pi ic}G(a, c) \tag{92}$$

$$G(a, c + 1) = G(a, c). \tag{93}$$

Condition (2-c) guarantees that $G(a, c) \in L^1[(0, 1), dc]$ for noninteger values of a , so that it has a Fourier expansion in the c -variable:

$$G(a, c) \sim \sum_{n \in \mathbb{Z}} g_n(a)e^{2\pi inc},$$

and condition (2-c) asserts that the $g_n(a)$ are continuous functions of a , for $0 < a < 1$. The twisted-periodicity condition (92) now implies that the Fourier coefficient functions $g_n(a)$ are defined for all $a \in \mathbb{R} \setminus \mathbb{Z}$ and satisfy

$$g_n(a + \ell) = g_{n-\ell}(a). \tag{94}$$

By hypothesis

$$e^{2\pi iac}T_m(F)(a, c) = e^{2\pi iac}(m^{-s}F(a, c)) = m^{-s}G(a, c)$$

holds on $(\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \frac{1}{m}\mathbb{Z})$. Thus we have the Fourier series

$$e^{2\pi iac}T_m(F)(a, c) \sim \sum_{n \in \mathbb{Z}} m^{-s}g_n(a)e^{2\pi inc}. \tag{95}$$

We evaluate the left side by expanding the Hecke operator and note that

$$G_k(a, c) := e^{2\pi iac}F\left(\frac{a+k}{m}, mc\right) = e^{-2\pi ikc}G\left(\frac{a+k}{m}, mc\right)$$

satisfies the twisted-periodicity conditions

$$G_k(a + m, c) = e^{2\pi imc}G_k(a, c) \tag{96}$$

$$G_k(a, c + 1) = G_k(a, c). \tag{97}$$

Condition (2-c) allows us to deduce that this function is in $L^1[(0, 1), dc]$, so it has a Fourier series expansion in the c -variable, which is

$$G_k(a, c) \sim e^{-2\pi ikc} \sum_{n \in \mathbb{Z}} g_n\left(\frac{a+k}{m}\right) e^{2\pi in(mc)}.$$

Summing up over $0 \leq k \leq m - 1$, we obtain

$$e^{2\pi iac} T_m(F)(a, c) \sim \frac{1}{m} \sum_{k=0}^{m-1} \left(\sum_{n \in \mathbb{Z}} g_n \left(\frac{a+k}{m} \right) e^{2\pi i(mn-k)c} \right).$$

By uniqueness of Fourier series of L^1 -functions, we obtain $m^{-s} g_{mn-k}(a) = \frac{1}{m} g_n \left(\frac{a+k}{m} \right)$, which we rewrite as

$$g_{mn-k}(a) = m^{s-1} g_n \left(\frac{a+k}{m} \right), \tag{98}$$

which is valid for $\frac{a+k}{m} \in \mathbb{R} \setminus \mathbb{Z}$. We now set

$$\tilde{g}_n(a) := g_n(a) |a - n|^{1-s},$$

and note that

$$\begin{aligned} \tilde{g}_{mn-k}(a) &= g_{mn-k}(a) |a - mn + k|^{1-s} = m^{s-1} g_n \left(\frac{a+k}{m} \right) |a - mn + k|^{1-s} \\ &= \tilde{g}_n \left(\frac{a+k}{m} \right). \end{aligned}$$

On choosing $n = k = 0$ we obtain for all $m \geq 1$ that

$$\tilde{g}_0(a) = \tilde{g}_0 \left(\frac{a}{m} \right) \tag{99}$$

is valid for $\frac{a}{m} \in \mathbb{R} \setminus \mathbb{Z}$. Now $g_0(a)$ is continuous on $(0, 1)$, so the left side of this equation implies that it extends to a continuous function on $(0, m)$ for all m , hence on $(0, \infty)$. Now $g_0(a - 1) = g_1(a)$ shows by condition (2-c) that $g_0(a)$ is continuous on $(-1, 0)$, and (99) for all m implies that $\tilde{g}(a)$ continuously extends to $(-\infty, 0)$. Since $\tilde{g}_0 \left(\frac{a}{m_1} \right) = \tilde{g}_0(m_2)$ for all positive integers m_1, m_2 , we conclude $\tilde{g}_0(a) = \tilde{g}_0(ra)$ for all positive rational r , which with the continuity conditions forces $\tilde{g}_0(a)$ to be constant on $(-\infty, 0)$ and on $(0, \infty)$, say $\tilde{g}_0(a) = A$ on $(0, \infty)$, resp. B on $(-\infty, 0)$. We deduce that

$$g_0(a) = \begin{cases} A|a|^{s-1} & \text{if } a > 0, \\ B|a|^{s-1} & \text{if } a < 0. \end{cases}$$

Now (94) gives

$$g_n(a) = g_0(a - n) = \begin{cases} A|a - n|^{s-1} & \text{if } a > n, \\ B|a - n|^{s-1} & \text{if } a < n. \end{cases}$$

Thus the Fourier series of $G(a, c)$ agrees term-by-term with the Fourier series of

$$H(a, c) := \frac{1}{2}(A + B)e^{-2\pi iac} L^+(1 - s, 1 - c, a) + \frac{1}{2}(A - B)e^{-2\pi iac} L^-(1 - s, 1 - c, a). \tag{100}$$

Since $\Re(s) < 1$ this function is in $L^1[(0, 1), dc]$ for noninteger a , and we may conclude by continuity that $G(a, c) = H(a, c) \in \mathcal{E}_s$, for $(a, c) \in (\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z})$. This completes the proof. \square

7 Concluding remarks

This paper studied two-variable Hecke operators T_m given by (6) on spaces of functions of two real variables. It showed that the Lerch zeta function $\zeta(s, a, c)$ is a simultaneous eigenfunction of all T_m with eigenvalue m^{-s} for all $s \in \mathbb{C}$. As mentioned in Sect. 1.2, we may formally define a *zeta operator* by

$$Z := \sum_{m=1}^{\infty} T_m.$$

For a fixed complex value $\Re(s) > 1$, we can make sense of this operator and observe that it has the Lerch zeta function as an eigenfunction and the Riemann zeta value $\zeta(s)$ as an eigenvalue, that is,

$$Z(\zeta)(s, a, c) = \zeta(s)\zeta(s, a, c).$$

In this paper, we have extended the action of the individual operators T_m to arbitrary complex values of s on suitable function spaces. In particular, one can define the individual T_m in the Hilbert space $L^2(\square, da dc)$ when $0 < \Re(s) < 1$. In this context, one may ask whether the other structures attached to the Lerch zeta function in these four papers can yield insight into the Riemann hypothesis.

Concerning the two-variable Hecke operators themselves and the Riemann hypothesis, we make the following observations.

- (1) At $a = 0$, the operator T_m degenerates to the dilation operator $\tilde{T}_m(f)(c) = f(mc)$. In 1999, Báez-Duarte [4] noted that this family of dilation operators relates to the real-variables approach to the Riemann hypothesis due to Nyman [33] and Beurling [11], see also Báez-Duarte [5], Burnol [14, 15] and Bagchi [6]. Therefore, one may ask whether there is a Riemann hypothesis criterion directly formulable in terms of the two-variable Hecke operators T_m .
- (2) The Lerch zeta function at $a = 0$ reduces for $\Re(s) > 1$ to the Hurwitz zeta function. The Hurwitz zeta function inherits the discontinuities of the Lerch zeta function at integer values of c . Milnor [32, p. 281] noted that at the value $s = 1$ the space \mathcal{K}_s includes on $(0, 1)$ the odd function $c - \frac{1}{2}$ which, due to the discontinuities, extends to the periodic function $\beta_1(c) := \{c - \frac{1}{2}\}$, the first Bernoulli polynomial, a fractional part function. The fractional part function appears in the various real-variables forms of the Riemann hypothesis above. The discontinuities of the Lerch zeta function at integer values of a or c (for some values of s) represent an important feature of these functions, worthy of further study in this context.

An interesting connection to the Riemann hypothesis relates to the differential operator $D_L = D_L^+ D_L^-$ considered here (and in [29, 30]) for which the Lerch zeta function is an eigenfunction.

- (1) It is natural to consider the *symmetrized Lerch differential operator*

$$\Delta_L := \frac{1}{2} (D_L^+ D_L^- + D_L^- D_L^+) = \frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + c \frac{\partial}{\partial c} + \frac{1}{2} \mathbf{I}. \tag{101}$$

This operator has many features of a Laplacian operator. It is formally skew-symmetric and satisfies

$$(\Delta_L \zeta)(s, a, c) = - \left(s - \frac{1}{2} \right) \zeta(s, a, c),$$

so that the line of skew symmetry is the critical line $Re(s) = \frac{1}{2}$. This operator has the “ xp ” form suggested by Berry and Keating [8,9], as the appropriate form for a “Hilbert–Polya” operator encoding the zeta zeros as eigenvalues.

- (2) The operator Δ_L commutes with all the T_m on the two-dimensional Lerch eigenspace \mathcal{E}_s , which is, however, not contained in $L^2(\square, da dc)$. It formally commutes with the T_m , but its commutativity depends on the specified domain of the unbounded operator Δ_L , viewed inside $L^2(\square, da dc)$. Such a domain is specified in [27, Sect. 9.2], for which the resulting operator Δ_L has purely continuous spectrum.
- (3) In order to view Δ_L as a suitable Hilbert–Polya operator for zeta zeros along these lines, it may be that one must instead find a scattering on \mathcal{H} and a small closed subspace of \mathcal{H} carrying the operator Δ_L . Related viewpoints on Hilbert–Polya operators have been proposed by Connes [16,17], Burnol [12] and the first author [26].

Author details

¹Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, USA, ²Department of Mathematics, Pennsylvania State University, University Park, PA 16802-8401, USA.

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