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# The “Riemann Hypothesis” is true for period polynomials of almost all newforms

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## Abstract

The period polynomial  $r_f(z)$  for a weight  $k \geq 3$  and level  $N$  newform  $f \in S_k(\Gamma_0(N), \chi)$  is the generating function for special values of  $L(s, f)$ . The functional equation for  $L(s, f)$  induces a functional equation on  $r_f(z)$ . Jin, Ma, Ono, and Soundararajan proved that for all newforms  $f$  of even weight  $k \geq 4$  and trivial nebentypus, the “Riemann Hypothesis” holds for  $r_f(z)$ : that is, all roots of  $r_f(z)$  lie on the circle of symmetry  $|z| = 1/\sqrt{N}$ . We generalize their methods to prove that this phenomenon holds for all but possibly finitely many newforms  $f$  of weight  $k \geq 3$  with any nebentypus. We also show that the roots of  $r_f(z)$  are equidistributed if  $N$  or  $k$  is sufficiently large.

## 1 Introduction and statement of results

Let  $f \in S_k(\Gamma_0(N), \chi)$  be a newform of weight  $k$ , level  $N$ , and nebentypus  $\chi$ . Associated with  $f$  is an  $L$ -function  $L(s, f)$ , which can be normalized so that the completed  $L$ -function

$$\Lambda(s, f) := N^{s/2} \int_0^\infty f(iy)y^{s-1}dy = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s, f) \quad (1.1)$$

satisfies the functional equation

$$\Lambda(s, f) = \epsilon(f) \Lambda(k - s, \tilde{f}) \quad (1.2)$$

for some  $\epsilon(f) \in \mathbb{C}$  with  $|\epsilon(f)| = 1$ .

The *period polynomial* associated with  $f$  is the degree  $k - 2$  polynomial defined by

$$r_f(z) := \int_0^{i\infty} f(y)(y - z)^{k-2}dy. \quad (1.3)$$

By the binomial theorem, we have

$$\begin{aligned} r_f(z) &= i^{k-1}N^{-\frac{k-1}{2}} \sum_{n=0}^{k-2} \binom{k-2}{n} (\sqrt{N}iz)^n \Lambda(k-1-n, f), \\ &= -\frac{(k-2)!}{(2\pi i)^{k-1}} \sum_{n=0}^{k-2} \frac{(2\pi iz)^n}{n!} L(k-1-n, f). \end{aligned}$$

Thus,  $r_f(z)$  is the generating function for the special values  $L(1, f), L(2, f), \dots, L(k-1, f)$  of the  $L$ -function associated with  $f$ . For background on period polynomials, we refer the reader to [1, 6, 7, 10, 11].

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When  $k \geq 3$ , the period polynomial  $r_f(z)$  is nonconstant, so one can consider where the roots of  $r_f(z)$  are located. To this end, we use the functional equation (1.2) to observe that

$$\overline{r_f(z)} = -(\sqrt{N}i\bar{z})^{k-2}\epsilon(f)^{-1}r_f\left(\frac{1}{N\bar{z}}\right).$$

Thus, if  $\rho$  is a root of  $r_f(z)$ , then  $\frac{1}{N\bar{\rho}}$  is also a root. Much like the behavior of the nontrivial zeroes of  $L(s, f)$  predicted by the generalized Riemann Hypothesis, one can consider whether all the roots of  $r_f(z)$  lie on the curve of symmetry of the roots: in this case, the circle  $|\rho| = 1/\sqrt{N}$ . It is natural to expect the following conjecture, which is supported by extensive numerical evidence.

**Conjecture** (“Riemann Hypothesis” for period polynomials) Let  $f \in S_k(\Gamma_0(N), \chi)$  be a newform. Then, the roots of  $r_f(z)$  all lie on the circle  $|\rho| = 1/\sqrt{N}$ .

El-Guindy and Raji [3] proved this for Hecke eigenforms on  $SL_2(\mathbb{Z})$  with full level ( $N = 1$ , for which the circle of symmetry is  $|z| = 1$ ). They were inspired by the work of Conrey et al. [2], who showed an analogous result for the odd parts of these period polynomials, again with full level.

Recent work by Jin et al. [5] proved the conjecture for all newforms of even weight  $k \geq 4$  and trivial nebentypus. They also showed that the roots of  $r_f(z)$  are equidistributed on the circle of symmetry for sufficiently large  $N$  or  $k$ . Using similar methods, Löbrich et al. [8] proved an analogous result for polynomials generating special values of  $L(s, \mathcal{M})$  for a sufficiently well-behaved class of motives  $\mathcal{M}$  with odd weight and even rank.

In this paper, we generalize the methods of [5] to prove the conjecture for all but possibly finitely many newforms.

**Theorem 1.1** *The “Riemann Hypothesis” for period polynomials holds for all but possibly finitely many newforms with weight  $k \geq 3$  and nontrivial nebentypus.*

*Remark* Note that for  $k < 3$ , the period polynomial is a constant. Therefore, Theorem 1.1 is essentially the best result for which one could hope, since an effective computation can check that Theorem 1.1 also holds for the finitely many possible exceptions. We denote the set of these finitely many newforms as  $\mathcal{S}$ , which consists of the following:

- (1) For  $k = 5$ , all newforms with level  $N \leq 10331$ .
- (2) For  $k \geq 6$ , all newforms with level  $N \leq C(k)$ , where  $C(k)$  is a constant given by tables at the end of Sects. 4 and 5.

We know of no counterexamples to Theorem 1.1.

We also show that the roots of  $r_f(z)$  are equidistributed on the circle of symmetry for sufficiently large  $N$  or  $k$ .

**Theorem 1.2** *Let  $f \in S_k(\Gamma_0(N), \chi)$  be a newform of weight  $k \geq 4$ , level  $N$ , and nebentypus  $\chi$  such that  $f \notin \mathcal{S}$ . Then, the following are true:*

- (i) *Suppose that  $k = 4$ , and let  $z_1, z_2$  denote the roots of  $r_f(z)$ . Then for any real  $\epsilon > 0$ ,*

$$\arg z_1 - \arg z_2 \equiv \pi + O_\epsilon\left(N^{-\frac{1}{4}+\epsilon}\right) \pmod{2\pi},$$

*where the implied constant depends only on  $\epsilon$  and is effectively computable.*

(ii) Suppose that  $k = 5$ . There exists  $c_f \in \mathbb{R}$  such that the arguments of the roots of  $r_f(z)$  can be written as

$$c_f + \theta_\ell + O_\varepsilon \left( \frac{1}{N^{\frac{1}{2}-\varepsilon}} \right) \pmod{2\pi}, \quad 0 \leq \ell \leq 2,$$

where  $\theta_\ell$  denotes the unique solution mod  $2\pi$  of

$$\frac{k-2}{2}\theta_\ell - \frac{2\pi}{\sqrt{N}} \sin \theta_\ell = \ell\pi,$$

and the implied constant depends only on  $\varepsilon$  and is effectively computable.

(iii) Suppose that  $k > 5$ . There exists  $c_f \in \mathbb{R}$  such that the arguments of the roots of  $r_f(z)$  can be written as

$$c_f + \theta_\ell + O \left( \frac{1}{2^{k/2}\sqrt{N}} \right) \pmod{2\pi}, \quad 0 \leq \ell \leq k-3,$$

Here,  $\theta_\ell$  is the unique solution mod  $2\pi$  to the equation

$$\frac{k-2}{2}\theta_\ell - \frac{2\pi}{\sqrt{N}} \sin \theta_\ell = \ell\pi,$$

and the implied constant is absolute and effectively computable.

In Sect. 2, we introduce notation and lemmas that we will be using in our proof. In Sect. 3, we will prove our main results for  $k = 3, 4$ , and  $5$  using ad hoc arguments. For larger  $k$ , we prove Theorem 1.1 in Sect. 4 (the case of  $k$  even) and Sect. 5 (the case of  $k$  odd), and we prove Theorem 1.2 in Sect. 6. Finally, in Sect. 7, we detail our Sage computations suggesting that the roots of the period polynomial of the newform

$$f(\tau) = q + 10q^3 + 64q^4 + 74q^5 + O(q^6) \in S_7 \left( \Gamma_0(11), \left( \begin{smallmatrix} -11 \\ \bullet \end{smallmatrix} \right) \right)$$

are all on the circle  $|z| = 1/\sqrt{11}$ . This newform  $f$  is in our finite set  $\mathcal{S}$  of possible exceptions, which suggests that Theorem 1.1 should be true even for newforms in  $\mathcal{S}$ .

## 2 Preliminaries

Throughout this section, we assume that  $f \in S_k(\Gamma_0(N), \chi)$  is a newform of weight  $k \geq 3$ , level  $N$ , and arbitrary nebentypus  $\chi$ . We note that the nebentypus character will be essentially invisible throughout our proof, other than the fact that it determines the level of  $f$ . We now define some notation related to  $r_f(z)$  and prove lemmas about the values of  $\Lambda(s, f)$  and  $L(s, f)$  along the real line. The lemmas will be very similar in spirit to those proven in [5].

Define  $\delta$  to satisfy  $\delta^2 = \epsilon(f)^{-1}$ . Now, define

$$t_f(z) = i^{-k+1} N^{\frac{k-1}{2}} z^{-\frac{k-2}{2}} \delta \cdot r_f \left( \frac{z}{i\sqrt{N}} \right) = \sum_{n=0}^{k-2} \binom{k-2}{n} z^{n-\frac{k-2}{2}} \delta \Lambda(k-1-n, f), \quad (2.1)$$

where  $z^{\frac{1}{2}}$  denotes  $r^{\frac{1}{2}} e^{\theta i/2}$  for  $z = r e^{\theta i}$  and  $0 \leq \theta < 2\pi$ . Using (1.2), one can compute

$$\overline{t_f(z)} = t_f \left( \frac{1}{\bar{z}} \right).$$

Therefore, if  $t_f(z) = 0$ , then  $t_f \left( \frac{1}{\bar{z}} \right) = 0$ . Additionally, for  $|z| = 1$ , we also have  $\overline{t_f(z)} = t_f(z)$ , so  $t_f(z)$  is real for  $|z| = 1$ . Note that  $t_f(z) = 0$  if and only if  $r_f \left( \frac{z}{i\sqrt{N}} \right) = 0$ . Therefore, to

prove Theorems 1.1 and 1.2, it suffices to show that all roots of  $t_f(z)$  lie on the circle  $|z| = 1$  and are equidistributed.

We will require the following monotonicity result.

**Lemma 2.1** *We have*

$$\left| \Lambda\left(\frac{k}{2}, f\right) \right| < \left| \Lambda\left(\frac{k}{2} + 1, f\right) \right| < \dots < \left| \Lambda\left(\frac{k}{2} + j, f\right) \right| < \dots$$

Also, for all  $0 < a < b$ ,

$$\left| \Lambda\left(\frac{k+1}{2} + a, f\right) \right| < \left| \Lambda\left(\frac{k+1}{2} + b, f\right) \right|.$$

*Proof* As  $\Lambda(s, f)$  is entire of order 1, we apply the Hadamard factorization theorem to write

$$\Lambda(s, f) = e^{A+Bs} \prod_{\rho} (1 - s/\rho) e^{s/\rho},$$

where the sum is taken over all roots  $\rho$  of  $\Lambda(s, f)$ . By [4, Proposition 5.7(3)], we have that

$$\operatorname{Re}(B) = - \sum_{\rho} \operatorname{Re}\left(\frac{1}{\rho}\right).$$

Note that  $\frac{k-1}{2} < \operatorname{Re}(\rho) < \frac{k+1}{2}$ . This implies that  $|1 - \frac{s}{\rho}|$  is increasing for  $s \geq \frac{k+1}{2}$  and  $|1 - \frac{k/2}{\rho}| < |1 - \frac{k/2+1}{\rho}|$ , from which the lemma follows.  $\square$

We also prove a useful inequality on ratios of  $L$ -function values.

**Lemma 2.2** *For all  $0 < a < b$ , we have*

$$\left| \frac{L\left(\frac{k+1}{2} + a, f\right)}{L\left(\frac{k+1}{2} + b, f\right)} - 1 \right| \leq \frac{\zeta(1+a)^2}{\zeta(1+b)^2} - 1.$$

*Proof* We have that

$$\left| \frac{L\left(\frac{k+1}{2} + a, f\right)}{L\left(\frac{k+1}{2} + b, f\right)} - 1 \right| = \left| \exp\left(-\int_a^b \frac{L'\left(\frac{k+1}{2} + s, f\right)}{L\left(\frac{k+1}{2} + s, f\right)} ds\right) - 1 \right|.$$

If we express

$$-\frac{L'(s, f)}{L(s, f)} = \sum \frac{\Lambda_f(n)}{n^s},$$

then Deligne’s bound on the eigenvalues of the Hecke operators on  $S_k(\Gamma_0(N), \chi)$  states that

$$|\Lambda_f(n)| \leq 2n^{\frac{k-1}{2}} \Lambda(n) \tag{2.2}$$

where  $\Lambda(n)$  denotes the von Mangoldt function; for a reference, see [9, Theorem 2.32].

Therefore, we have that

$$\left| \int_a^b \frac{L'\left(\frac{k+1}{2} + s, f\right)}{L\left(\frac{k+1}{2} + s, f\right)} ds \right| \leq -2 \int_a^b \frac{\zeta'(1+s)}{\zeta(1+s)} ds = 2 \log \frac{\zeta(1+a)}{\zeta(1+b)}. \tag{2.3}$$

Now the lemma follows from the inequality  $|e^x - 1| \leq e^{|x|} - 1$ .  $\square$

Finally, we show a lemma that serve as our main means of proving Theorem 1.1 for period polynomials.

**Lemma 2.3** *Let  $\text{sgn}(r)$  equal  $-1$  for negative real numbers  $r$ ,  $1$  for positive real numbers  $r$ , and  $0$  for  $r = 0$ . If there exist real numbers  $0 \leq \theta_1 < \theta_2 < \dots < \theta_k < 2\pi$  such that either*

$$\text{sgn}(t_f(e^{i\theta_j})) = (-1)^j \quad \text{for all } 1 \leq j \leq k,$$

or

$$\text{sgn}(t_f(e^{i\theta_j})) = (-1)^{j+1} \quad \text{for all } 1 \leq j \leq k,$$

then all solutions to  $t_f(z) = 0$  satisfy  $|z| = 1$ .

*Proof* First,  $t_f(z)$  is real for  $|z| = 1$ , so  $\text{sgn}(t_f(e^{i\theta_j}))$  is well defined. Now, by the intermediate value theorem, there exist  $\theta \in (\theta_j, \theta_{j+1})$  such that  $t_f(e^{i\theta}) = 0$  for all  $1 \leq j \leq k - 1$ . This gives us  $k - 1$  roots of  $t_f(z)$  that lie on  $|z| = 1$ . When  $k$  is even, we also get a root in the range  $(\theta_k, \theta_1 + 2\pi)$  by the intermediate value theorem. When  $k$  is odd, we may redefine the square root in order to move the discontinuity into an interval outside of  $[\theta_k, \theta_1 + 2\pi]$ . This would only affect the sign of  $t_f(z)$ . By the intermediate value theorem, this shows the existence of a zero with argument in the range  $[\theta_k, \theta_1 + 2\pi]$  as desired. As  $t_f(z) = 0$  for at most  $k$  values of  $z$ , the above argument shows that we have found all of them.  $\square$

### 3 Proof for weights 3, 4, and 5

Here we prove Theorems 1.1 and 1.2 for  $k = 3, 4$ , and  $5$ .

#### 3.1 The weight 3 case

For  $k = 3$ , (2.1) gives that

$$t_f(z) = \delta z^{-\frac{1}{2}} (\Lambda(2, f) + z\Lambda(1, f)).$$

By (1.2), we know that

$$|\Lambda(2, f)| = |\Lambda(1, f)|,$$

so the root of  $t_f(z)$  lies on the unit circle.

#### 3.2 The weight 4 case

For  $k = 4$ , we have

$$t_f(z) = \delta (z^{-1}\Lambda(3, f) + 2\Lambda(2, f) + z\Lambda(1, f)).$$

Now, note that for  $|z| = 1$ , it follows that

$$\frac{1}{2}t_f(z) = \text{Re}(\delta(\Lambda(2, f) + z\Lambda(1, f))) = \text{Re}(\delta\Lambda(2, f)) + \text{Re}(\delta z\Lambda(1, f)).$$

By Lemma 2.1, we have that

$$|\text{Re}(\delta\Lambda(2, f))| \leq |\Lambda(2, f)| < |\Lambda(3, f)| = |\Lambda(1, f)|,$$

so there exist 2 values of  $z$  with  $|z| = 1$  such that

$$\text{Re}(\delta z\Lambda(1, f)) = -\text{Re}(\delta\Lambda(2, f)),$$

as desired.

In order to prove Theorem 1.2(iii), we need to bound  $|\Lambda(2,f)|/|\Lambda(1,f)|$ . First, note that

$$|\Lambda(1,f)| = |\Lambda(3,f)| \gg N^{\frac{3}{2}}.$$

In order to bound  $|\Lambda(2,f)|$ , we appeal to the Phragmén–Lindelöf principle; specifically, see [4, Lemma 5.2, Theorem 5.53] and apply (2.2). This allows to obtain for any  $\epsilon > 0$

$$|\Lambda(2,f)| \leq \max_{t \in \mathbb{R}} |\Lambda(5/2 + \epsilon + it, f)| = \max_{t \in \mathbb{R}} N^{\frac{5}{4} + \frac{1}{2}\epsilon} |L(5/2 + \epsilon + it, f)| \ll_{\epsilon} N^{\frac{5}{4} + \epsilon},$$

Thus, we have that

$$\frac{|\Lambda(2,f)|}{|\Lambda(1,f)|} \ll N^{-\frac{1}{4} + \epsilon}$$

and the values of  $z$  satisfying  $t_f(z) = 0$  satisfy

$$\arg z = \pm \frac{\pi}{2} + \arg(\delta \Lambda(1,f)) + O(N^{-\frac{1}{4} + \epsilon})$$

### 3.3 The weight 5 case

For  $k = 5$ , we have

$$t_f(z) = \delta \left( z^{-\frac{3}{2}} \Lambda(4,f) + 3z^{-\frac{1}{2}} \Lambda(3,f) + 3z^{\frac{1}{2}} \Lambda(2,f) + z^{\frac{3}{2}} \Lambda(1,f) \right).$$

Once again, for  $|z| = 1$ , we have

$$\frac{1}{2} t_f(z) = \operatorname{Re} \left( \delta \left( 3z^{\frac{1}{2}} \Lambda(2,f) + z^{\frac{3}{2}} \Lambda(1,f) \right) \right) = \operatorname{Re}(3\delta z^{\frac{1}{2}} \Lambda(2,f)) + \operatorname{Re} \left( \delta z^{\frac{3}{2}} \Lambda(1,f) \right).$$

There exist three reals  $0 \leq \theta_1 < \theta_2 < \theta_3 < 2\pi$  such that  $|\operatorname{Re}(\delta(e^{i\theta_j})^{\frac{3}{2}} \Lambda(1,f))| = |\Lambda(1,f)|$  for  $1 \leq j \leq 3$ , and  $\operatorname{Re}(\delta(e^{i\theta_j})^{\frac{3}{2}} \Lambda(1,f))$  alternates in sign. Thus, by Lemma 2.3, we are done if we are able to show that

$$|\operatorname{Re}(3\delta z^{\frac{1}{2}} \Lambda(2,f))| < |\Lambda(1,f)|,$$

which is equivalent to proving

$$\frac{|\Lambda(3,f)|}{|\Lambda(4,f)|} < \frac{1}{3}.$$

Let  $0 < \epsilon < 1$ . By Lemmas 2.1 and 2.2, it follows that

$$\begin{aligned} \frac{|\Lambda(3,f)|}{|\Lambda(4,f)|} &\leq \frac{|\Lambda(3 + \epsilon, f)|}{|\Lambda(4, f)|} = \frac{|L(3 + \epsilon, f)|}{|L(4, f)|} \left( \frac{2\pi}{\sqrt{N}} \right)^{1-\epsilon} \frac{\Gamma(3 + \epsilon)}{\Gamma(4)} \\ &\leq \frac{\zeta(1 + \epsilon)^2}{\zeta(2)^2} \left( \frac{2\pi}{\sqrt{N}} \right)^{1-\epsilon} \frac{\Gamma(3 + \epsilon)}{\Gamma(4)}. \end{aligned}$$

Choosing  $\epsilon = 2/5$ , the last expression is less than  $\frac{1}{3}$  for  $N \geq 10,332$ , which completes the proof for  $k = 5$ .

To show the desired equidistribution property, define  $\theta_1$  and  $\theta_2$  as above. Now let  $\theta_{\pm} = (\theta_1 + \theta_2)/2 \pm \epsilon$ , for  $\epsilon > 0$  to be chosen later. Then, we see that

$$|\operatorname{Re}(\delta(e^{i\theta_{\pm}})^{\frac{3}{2}} \Lambda(1,f))| = |\Lambda(1,f)| \sin \left( \frac{3}{2}\epsilon \right)$$

with the sign of  $\operatorname{Re}(\delta(e^{i\theta_{\pm}})^{\frac{3}{2}} \Lambda(1,f))$  being different for  $\epsilon > 0$  and  $\epsilon < 0$ . If we can show that

$$|\operatorname{Re}(3\delta z^{\frac{1}{2}} \Lambda(2,f))| \leq \left| \sin \left( \frac{3}{2}\epsilon \right) \right| \cdot |\Lambda(1,f)|$$

then Lemma 2.3 will show that the root has an argument lying between  $\theta_-$  and  $\theta_+$ . By the bounding above, we only require

$$\frac{\zeta(1 + \epsilon)^2}{\zeta(2)^2} \left(\frac{2\pi}{\sqrt{N}}\right)^{1-\epsilon} \frac{\Gamma(3 + \epsilon)}{\Gamma(4)} < \frac{1}{3} \sin\left(\frac{3}{2}\epsilon\right).$$

Choosing  $\epsilon = O(N^{-\frac{1}{2}+\epsilon})$  suffices.

#### 4 Proof for remaining even weights

In this section, we will show Theorem 1.1 for all even weights  $k \geq 6$ . Throughout the section, we will restrict our attention to those  $z$  such that  $|z| = 1$ . For simplicity, let  $m = \frac{k-2}{2}$ , and define

$$\begin{aligned} P_f(z) &= \frac{1}{2} \binom{2m}{m} \Lambda(m+1, f) + \sum_{n=0}^{m-1} \binom{k-2}{n} z^{m-n} \delta \Lambda(n+1, f) \\ &= \frac{1}{2} \binom{2m}{m} \Lambda(m+1, f) \\ &\quad + (2m)! \left(\frac{\sqrt{N}}{2\pi}\right)^{2m+1} \delta^{-1} z^m \sum_{n=0}^{m-1} \frac{1}{n!} \left(\frac{2\pi}{z\sqrt{N}}\right)^n \overline{L(2m+1-n, f)}. \end{aligned} \tag{4.1}$$

This satisfies

$$t_f(z) = P_f(z) + \overline{P_f(z)} = 2 \operatorname{Re}(P_f(z)).$$

Next, define

$$\begin{aligned} Q_f(z) &= \frac{1}{(2m)!} \left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} P_f(z) \\ &= \frac{1}{2(m!)^2} \left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} \Lambda(m+1, f) \\ &\quad + \delta^{-1} z^m \sum_{n=0}^{m-1} \frac{1}{n!} \left(\frac{2\pi}{z\sqrt{N}}\right)^n \overline{L(2m+1-n, f)}. \end{aligned}$$

Note that  $\operatorname{sgn}(Q_f(z)) = \operatorname{sgn}(P_f(z))$ . As in [5], rewrite

$$Q_f(z) = \delta^{-1} \overline{L(2m+1, f)} z^m \exp\left(\frac{2\pi}{z\sqrt{N}}\right) + S_1(z) + S_2(z) + S_3(z),$$

where we define

$$\begin{aligned} S_1(z) &= \delta^{-1} \overline{L(2m+1, f)} z^m \sum_{n=0}^{m-1} \frac{1}{n!} \left(\frac{2\pi}{z\sqrt{N}}\right)^n \left(\frac{\overline{L(2m+1-n, f)}}{L(2m+1, f)} - 1\right) \\ S_2(z) &= -\delta^{-1} \overline{L(2m+1, f)} z^m \sum_{n \geq m} \frac{1}{n!} \left(\frac{2\pi}{z\sqrt{N}}\right)^n \\ S_3(z) &= \frac{1}{2(m!)^2} \left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} \Lambda(m+1, f). \end{aligned}$$

For  $z = e^{i\theta}$ , note that

$$\arg\left(\delta^{-1} \overline{L(2m+1, f)} z^m \exp\left(\frac{2\pi}{z\sqrt{N}}\right)\right) = C + m\theta - \frac{2\pi \sin \theta}{\sqrt{N}}, \tag{4.2}$$

where  $C$  is a fixed constant depending on  $\delta$  and  $L(2m+1, f)$ . Therefore, we can pick  $k$  values of  $z$  on the circle  $|z| = 1$  such that the previous expression has argument  $\ell\pi$  for

integers  $\ell$ . The value of  $Q_f(z)$  at these points has alternating positive and negative real part with magnitude at least  $|L(2m + 1, f)| \exp\left(-\frac{2\pi}{\sqrt{N}}\right)$ . By Lemma 2.3, it suffices to show that

$$|S_1(z)| + |S_2(z)| + |S_3(z)| < |L(2m + 1, f)| \exp\left(-\frac{2\pi}{\sqrt{N}}\right).$$

To bound  $S_1(z)$ , we use Lemma 2.2 in the form  $\left|\frac{L(2m+1-n, f)}{L(2m+1, f)}\right| \leq \zeta\left(\frac{1}{2} + m - n\right)^2 - 1$ . This gives

$$|S_1(z)| \leq |L(2m + 1, f)| \sum_{n=1}^{m-1} \frac{1}{n!} \left(\frac{2\pi}{\sqrt{N}}\right)^n (\zeta(1/2 + m - n)^2 - 1).$$

For the term  $n = m - 1$  in the above expression, we use the bound  $\zeta(3/2)^2 - 1 \leq 35/6$ . For  $0 \leq n \leq m - 2$ , note that  $2^x(\zeta(1/2 + x)^2 - 1)$  is decreasing for  $x \geq 2$ . Therefore, for  $0 \leq n \leq m - 2$ , we find that

$$\zeta(1/2 + m - n)^2 - 1 \leq 2^{n-m} \cdot 4(\zeta(5/2)^2 - 1) \leq \frac{16}{5} 2^{n-m}.$$

Now, we combine the above estimates with  $S_2(z)$  to obtain

$$\begin{aligned} \frac{|S_1(z)| + |S_2(z)|}{|L(2m + 1, f)|} &\leq \frac{16}{5} \sum_{n=1}^{m-1} \frac{1}{n!} \left(\frac{2\pi}{\sqrt{N}}\right)^n \frac{2^n}{2^m} + \frac{17}{4} \frac{1}{(m-1)!} \left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} \\ &\quad + \sum_{n \geq m} \frac{1}{n!} \left(\frac{2\pi}{\sqrt{N}}\right)^n \\ &\leq \frac{16}{5} 2^{-m} \left(\exp\left(\frac{4\pi}{\sqrt{N}}\right) - 1\right) + \frac{17}{4} \frac{1}{(m-1)!} \left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}. \end{aligned} \tag{4.3}$$

To finish, we estimate  $|S_3(z)|$  using Lemma 2.1 and then 2.2.

$$\begin{aligned} |S_3(z)| &\leq \frac{1}{2(m!)^2} \left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} |\Lambda(m + 1, f)| \leq \frac{1}{2(m!)^2} \left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} |\Lambda(m + 2, f)| \\ &\leq \frac{m + 1}{2m!} \left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} |L(m + 2, f)| \\ &\leq \frac{m + 1}{2m!} \left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} |L(2m + 1, f)| \zeta(3/2)^2 \\ &\leq \frac{7}{2} \frac{m + 1}{m!} \left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} |L(2m + 1, f)|. \end{aligned} \tag{4.4}$$

By using (4.3) and (4.4), it suffices to verify

$$\begin{aligned} &\frac{16}{5} 2^{-m} \left(\exp\left(\frac{4\pi}{\sqrt{N}}\right) - 1\right) + \frac{17}{4} \frac{1}{(m-1)!} \left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} + \frac{7}{2} \frac{m + 1}{m!} \left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} \\ &< \exp\left(-\frac{2\pi}{\sqrt{N}}\right). \end{aligned} \tag{4.5}$$

For each value of  $m$  in the first row on the following table, the value  $N(m)$  is such that inequality (4.5) holds for all  $N \geq N(m)$ . Note that the case  $m = 1$  was done in Sect. 3.

$m$	29	21	18	16	14	13	12	11	10	9	8	7	6	5	4	3	2
$N(m)$	1	2	3	4	5	6	7	9	11	14	19	27	41	69	142	433	5875

Therefore, for all  $m \geq 29$ ,  $N(m) = 1$ . This completes our proof of Theorem 1.1 for  $k$  even.



### 5 Proof for remaining odd weights

In this section, we will show Theorem 1.1 for all odd weights  $k \geq 7$ . As in the above section, we will restrict our attention to those  $z$  such that  $|z| = 1$ . For simplicity, let  $m = \frac{k-3}{2}$ , and define

$$\begin{aligned}
 P_f(z) &= \sum_{n=0}^m \binom{k-2}{n} z^{m-n+\frac{1}{2}} \delta \Lambda(n+1, f) \\
 &= (2m+1)! \left(\frac{\sqrt{N}}{2\pi}\right)^{2m+2} \delta^{-1} z^{m+\frac{1}{2}} \sum_{n=0}^m \frac{1}{n!} \left(\frac{2\pi}{z\sqrt{N}}\right)^n \overline{L(2m+2-n, f)}, \quad (5.1)
 \end{aligned}$$

so  $t_f(z) = 2 \operatorname{Re}(P_f(z))$ . As in the above section, define

$$\begin{aligned}
 Q_f(z) &= \frac{1}{(2m+1)!} \left(\frac{2\pi}{\sqrt{N}}\right)^{2m+2} P_f(z) = \delta^{-1} z^{m+\frac{1}{2}} \sum_{n=0}^m \frac{1}{n!} \left(\frac{2\pi}{z\sqrt{N}}\right)^n \overline{L(2m+2-n, f)} \\
 &= \overline{L(2m+2, f)} \delta^{-1} z^{m+\frac{1}{2}} \exp\left(\frac{2\pi}{z\sqrt{N}}\right) + S_1(z) + S_2(z) + S_3(z),
 \end{aligned}$$

where  $S_1(z)$ ,  $S_2(z)$ , and  $S_3(z)$  are defined as follows.

$$\begin{aligned}
 S_1(z) &= \overline{L(2m+2, f)} \delta^{-1} z^{m+\frac{1}{2}} \sum_{n=0}^{m-1} \frac{1}{n!} \left(\frac{2\pi}{z\sqrt{N}}\right)^n \left(\frac{\overline{L(2m+2-n, f)}}{L(2m+2, f)} - 1\right), \\
 S_2(z) &= -\overline{L(2m+2, f)} \delta^{-1} z^{m+\frac{1}{2}} \sum_{n \geq m} \frac{1}{n!} \left(\frac{2\pi}{z\sqrt{N}}\right)^n, \\
 S_3(z) &= \delta^{-1} z^{m+\frac{1}{2}} \frac{1}{m!} \left(\frac{2\pi}{z\sqrt{N}}\right)^m \overline{L(m+2, f)}.
 \end{aligned}$$

As in Sect. 4, it suffices to show that

$$|S_1(z)| + |S_2(z)| + |S_3(z)| < |L(2m+2, f)| \exp\left(-\frac{2\pi}{\sqrt{N}}\right).$$

The proof of this will proceed in a very similar way to that of the above section. Note that the function  $2^x(\zeta(1+x)^2 - 1)$  is decreasing for  $x \geq 1$ . By Lemma 2.2, for  $0 \leq n \leq m-1$ , we can bound

$$\left| \frac{L(2m+2-n, f)}{L(2m+2, f)} - 1 \right| \leq \zeta(1+m-n)^2 - 1 \leq 2^{n-m} \cdot 2(\zeta(2)^2 - 1) \leq 2^{n-m} \cdot \frac{7}{2}.$$

By Lemma 2.2, we have

$$\begin{aligned}
 \frac{|S_1(z)| + |S_2(z)|}{|L(2m+2, f)|} &\leq \frac{7}{2} 2^{-m} \sum_{n=1}^{m-1} \frac{1}{n!} \left(\frac{4\pi}{\sqrt{N}}\right)^n + \sum_{n \geq m} \frac{1}{n!} \left(\frac{2\pi}{\sqrt{N}}\right)^n \frac{2^n}{2^m} \\
 &\leq \frac{7}{2} 2^{-m} \left(\exp\left(\frac{4\pi}{\sqrt{N}}\right) - 1\right). \quad (5.2)
 \end{aligned}$$

Now we use Lemma 2.1 to bound  $|L(m+2, f)|$ .

$$\begin{aligned}
 |L(m+2, f)| &\leq \frac{1}{(m+1)!} \left(\frac{2\pi}{\sqrt{N}}\right)^{m+2} |\Lambda(m+2, f)| \\
 &\leq (m+2) \left(\frac{2\pi}{\sqrt{N}}\right)^{-1} |L(m+3, f)|.
 \end{aligned}$$

Therefore, we have that

$$\begin{aligned}
 |S_3(z)| &\leq \frac{m+2}{m!} \left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} |L(m+3, f)| \\
 &\leq \frac{m+2}{m!} \left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} \zeta(2)^2 |L(2m+2, f)|,
 \end{aligned}
 \tag{5.3}$$

after applying Lemma 2.2. Finally, by using (5.2) and (5.3), it suffices to show that

$$\frac{7}{2} 2^{-m} \left(\exp\left(\frac{4\pi}{\sqrt{N}}\right) - 1\right) + \frac{m+2}{m!} \left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} \zeta(2)^2 < \exp\left(-\frac{2\pi}{\sqrt{N}}\right).
 \tag{5.4}$$

For each value of  $m$  in the first row on the following table, the value  $N(m)$  is such that inequality (5.4) holds for all  $N \geq N(m)$ . Note that the cases  $m = 0, 1$  were done in Sect. 3.

$m$	31	23	19	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2
$N(m)$	1	2	3	4	5	6	7	8	10	13	16	22	31	47	76	137	285	766	5258

Therefore, for  $m \geq 31, N(m) = 1$ . This completes the proof of Theorem 1.1 for  $k$  odd.

### 6 Equidistribution of roots for large weights

Let  $k \geq 6$ . For even  $k$ , set  $m = \frac{k-2}{2}$ . Then, the arguments in the previous sections show that for  $z = e^{i\theta}$ , then

$$Q_f(z) = |L(2m+1, f)| \left(\exp\left(i(m\theta + C) + \frac{2\pi}{\sqrt{N}} e^{-i\theta}\right) + O\left(\frac{1}{2^m \sqrt{N}}\right)\right)$$

for some real constant  $C$  is defined in (4.2). Therefore,

$$\begin{aligned}
 \operatorname{Re}(Q_f(z)) &= |L(2m+1, f)| \left(\exp\left(\frac{2\pi}{\sqrt{N}} \cos \theta\right) \cos\left(m\theta + C - \frac{2\pi}{\sqrt{N}} \sin \theta\right) \right. \\
 &\quad \left. + O\left(\frac{1}{2^m \sqrt{N}}\right)\right).
 \end{aligned}$$

Consider the  $\theta_\ell$  such that  $m\theta_\ell + C - \frac{2\pi}{\sqrt{N}} \sin \theta_\ell = \frac{\pi}{2} + \ell\pi$ . Then it is simple to verify that for some constant  $D$ , for the two values  $\theta_\ell \pm \frac{D}{2^m \sqrt{N}}$ ,  $\operatorname{Re}(Q_f(z))$  has different signs. This completes the proof for even  $k$ .

For odd  $k$ , set  $m = \frac{k-3}{2}$ . Then, the arguments in the previous section show that for  $z = e^{i\theta}$ ,

$$Q_f(z) = |L(2m+2, f)| \left(\exp\left(i\left(m + \frac{1}{2}\right)\theta + C\right) + \frac{2\pi}{\sqrt{N}} e^{-\theta i}\right) + O\left(\frac{1}{2^m \sqrt{N}}\right).$$

Therefore, it follows that

$$\begin{aligned}
 \operatorname{Re}(Q_f(z)) &= |L(2m+2, f)| \left(\exp\left(\frac{2\pi}{\sqrt{N}} \cos \theta\right) \cos\left(\left(m + \frac{1}{2}\right)\theta + C - \frac{2\pi}{\sqrt{N}} \sin \theta\right) \right. \\
 &\quad \left. + O\left(\frac{1}{2^m \sqrt{N}}\right)\right).
 \end{aligned}$$

Now, consider the values  $\theta_\ell$  such that  $(m + 1/2)\theta_\ell + C - \frac{2\pi}{\sqrt{N}} \sin \theta_\ell = \frac{\pi}{2} + \ell\pi$ . Once again, one can verify that for the values  $\theta_\ell \pm \frac{D}{2^m \sqrt{N}}$ ,  $\operatorname{Re}(Q_f(z))$  has opposite signs. This completes the proof of Theorem 1.2.

## 7 A numerical example

Consider the newform  $f \in S_7(\Gamma_0(11), (\frac{-11}{\bullet}))$  whose  $q$ -series is given by

$$q + 10q^3 + 64q^4 + 74q^5 + O(q^6).$$

All the coefficients of  $f$  are real, and we have  $\epsilon(f) = 1$ . In light of the functional equation  $L(s, f) = L(k - 1 - s, f)$ , we can use Sage to compute the critical values of  $L(s, f)$  and thereby obtain  $r_f(z)$ . We calculate that the roots of  $r_f(z)$  are

$$z_1 \approx -0.294570496142963 - 0.0643219535709181i,$$

$$z_2 \approx -0.204098252273756 + 0.221930156418385i,$$

$$z_3 \approx 0.301511344577764i,$$

$$z_4 \approx 0.204098252273756 + 0.221930156418385i,$$

and

$$z_5 \approx 0.294570496142963 - 0.0643219535709181i.$$

All five roots have absolute value  $\approx 0.301511344577764 \approx 1/\sqrt{11}$ , as expected given the statement of Theorem 1.1.

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