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On the regulator formulas of Bertolini, Darmon and Rotger

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To the memory of Robert Coleman.

Abstract

We give a unified, and somewhat simplified, account of the regulator formulas appearing in papers of Bertolini, Darmon and Rotger, describing the syntomic regulator on the first, second and third self-products of modular curves in terms of p-adic modular forms.

1 Introduction

In a recent sequence of papers [2,4,11] Bertolini, Darmon and Rotger relate, in several different situations, the syntomic regulator applied to some arithmetic object, to special values of p-adic L-functions. In particular, they use the author's previous work [6,8,9] to find certain formulas for this regulator, which are later related to p-adic L-values. In this short note, we attempt to provide what is hopefully a simpler proof of these regulator formulas. We hope this will prove beneficial for future work in this subject, extending the results to bad reduction situations. We have also attempted to make the presentation fairly self-contained.

Let us briefly summarize the difference between our techniques and those of the above-mentioned papers. In the above triad, the essential computation is done in the case of diagonal cycles on the triple product of modular curves in [11]. The fundamental trick is to eliminate the choice of a lift in finite polynomial cohomology of a certain class in de Rham cohomology by applying a certain correspondence. This idea might have other applications in the theory. In the 3 cases discussed here, the cohomology is sufficiently simple though that one can compute the lift directly, avoiding the need for a correspondence and in practice simplifying the argument. It also makes the 3 proofs very similar in structure, rather than making one proof dependent on the others. We have also made use of properties of the cup product in finite polynomial cohomology to further simplify the argument in [11].

I would like to thank Andreas Langer for a most enjoyable visit in Exeter and for the discussion on [11] which resulted in the present paper, as well as for reading the manuscript and carefully correcting some, but no doubt not all, of my errors. I would also like to thank Rob de Jeu, whose insistence encouraged me to figure out some of the results in Sect. 5. I would also like to thank the referee for finding at least some of my many errors in the previous version of this text and for making many comments that improved the readability of the paper. The bulk of the research described here was done when I was visiting the Mathematical Institute at the university of Oxford. I would like to thank the Institute and especially Alan Lauder and Minhyong Kim for the supporting my visit. This work was partially supported by ERC Grant Number 204083.

This paper is dedicated to the memory of Robert Coleman. The influence of Robert's ideas on my work is obvious. In particular, the theory of fp-cohomology is just an extension of Robert's idea of using polynomials in Frobenius to kill off cohomology on the way to obtaining his integrals. His presence will be greatly missed.

2 The setup

Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and residue field κ of size q.

We briefly recall some facts about finite polynomial cohomology as defined in [6]. This theory assigns to any smooth $\mathcal{O}_K$-scheme $\mathcal{X}$ a "cohomology group" $H^i_{\mathrm{fp}}(\mathcal{X}, n)$. More generally, for a finite set of weights $R \subset \mathbb{Z}_{\geq 0}$ it gives a group $H^i_{\mathrm{fp},R}(\mathcal{X}, n)$ in such a way that $H^i_{\mathrm{fp}}(\mathcal{X}, n) = H^i_{\mathrm{fp},\{2i\}}(\mathcal{X}, n)$. A basic property of this theory is that if R contains none of the weights on $H^{i-1}_{\mathrm{rig}}(\mathcal{X}_\kappa/K)$, where H_{rig} denotes Berthelot's rigid cohomology [5] and where $\mathcal{X}_\kappa$ is the special fiber of $\mathcal{X}$, this sits in a short exact sequence

$$0 \to H^{i-1}_{\mathrm{rig}}(\mathcal{X}_\kappa/K)/F^n H^{i-1}_{\mathrm{dR}}(X/K) \xrightarrow{\boldsymbol{i}} H^i_{\mathrm{fp},R}(\mathcal{X}, n) \xrightarrow{\boldsymbol{p}} F^n H^i_{\mathrm{dR}}(X/K)^R \to 0$$

where X denotes the generic fiber of $\mathcal{X}$, F^n refers to the n-th step in the Hodge filtration on the de Rham cohomology group H_{dR} and the superscript R indicates the part of de Rham cohomology mapping in rigid cohomology to the part having weights in R. This is true in particular when $\mathcal{X}$ is proper and $R = \{2i\}$, so that the sequence above reads (as also in this case $H_{\mathrm{dR}} = H_{\mathrm{rig}}$)

$$0 \to H^{i-1}_{\mathrm{dR}}(X/K)/F^n H^{i-1}_{\mathrm{dR}}(X/K) \xrightarrow{\boldsymbol{i}} H^i_{\mathrm{fp}}(\mathcal{X}, n) \xrightarrow{\boldsymbol{p}} F^n H^i_{\mathrm{dR}}(X/K) \to 0, \tag{2.1}$$

and it is also true when $\mathcal{X}$ has relative dimension 1 and $R = \{1, 2\}$ and $i = 1$. When $\mathcal{X}$ is proper and of relative dimension 1, and the generic fiber is irreducible, we specialize the above sequence in the following two cases

$$0 \to K \xrightarrow{\boldsymbol{i}} H^1_{\mathrm{fp}}(\mathcal{X}, 1) \xrightarrow{\boldsymbol{p}} F^1 H^1_{\mathrm{dR}}(X/K) \to 0 \tag{2.2}$$

$$\boldsymbol{p} : H^1_{\mathrm{fp}}(\mathcal{X}, 0) \xrightarrow{\sim} H^1_{\mathrm{dR}}(X/K). \tag{2.3}$$

There is a functorial cup product

$$H^{i_1}_{\mathrm{fp},R_1}(\mathcal{X}, n_1) \times H^{i_2}_{\mathrm{fp},R_2}(\mathcal{X}, n_2) \to H^{i_1+i_2}_{\mathrm{fp},R_1+R_2}(\mathcal{X}, n_1 + n_2),$$

which is compatible with cup products on de Rham and rigid cohomology [6, Proposition 2.5]. Here $R_1 + R_2$ means the sum of the sets of weights. In the proper case, the compatibility means that we have

$$\boldsymbol{p}(x \cup y) = \boldsymbol{p}(x) \cup \boldsymbol{p}(y), \quad x \in H^i_{\mathrm{fp}}(\mathcal{X}, n),\ y \in H^j_{\mathrm{fp}}(\mathcal{X}, m) \tag{2.4}$$

and

$$\boldsymbol{i}(x) \cup y = \boldsymbol{i}(x \cup \boldsymbol{p}(y)), \quad x \in H^{i-1}_{\mathrm{dR}}(X/K)/F^n,\ y \in H^j_{\mathrm{fp}}(\mathcal{X}, m), \tag{2.5}$$

where $\cup_{\mathrm{dR}}$ is the usual cup product in de Rham cohomology. We can record the following consequences of this.

Proposition 2.1 *Suppose $\mathcal{X}$ is proper of relative dimension* 1. *Let $\tilde{\omega}_i \in H^1_{\mathrm{fp}}(\mathcal{X}, 1)$ for $i = i, 2$ and $\eta \in H^1_{dR}(X/K)$. Set $\boldsymbol{p}\tilde{\omega}_i = \omega_i \in \Omega^1(X/K)$ and identify η with an element in $H^1_{\mathrm{fp}}(\mathcal{X}, 0)$ via the isomorphism $\boldsymbol{p}$ of* (2.3).

(1) *We have $\tilde{\omega}_1 \cup \tilde{\omega}_2 \in \boldsymbol{i}H^1_{dR}(X/K)$ and*

$$\eta \cup \tilde{\omega}_1 \cup \tilde{\omega}_2 = \boldsymbol{i}(\eta \cup_{dR} \boldsymbol{i}^{-1}(\tilde{\omega}_1 \cup \tilde{\omega}_2)).$$

(2) *For $\alpha \in K$, we have*

$$\eta \cup \boldsymbol{i}(\alpha) \cup \tilde{\omega}_1 = -\eta \cup \tilde{\omega}_1 \cup \boldsymbol{i}(\alpha) = \boldsymbol{i}(\alpha\eta \cup_{dR} \omega_1).$$

Proof We have $\boldsymbol{p}(\tilde{\omega}_1 \cup \tilde{\omega}_2) = \omega_1 \cup \omega_2 = 0$ by (2.4), as both ω_i are in Ω^1, hence $\tilde{\omega}_1 \cup \tilde{\omega}_2$ is in the image of $\boldsymbol{i}$ by the exact sequence (2.1). As, in our identification $\boldsymbol{p}(\eta) = \eta$, we can apply (2.5) to obtain the first formula. The second formula follows again from (2.5). □

We will need to recall some more details about fp-cohomology theory in the case that $\mathcal{X}$ is affine and of relative dimension 1 (some things here hold in greater generality but we attempt to make things as concrete as possible). We may associate with the special fiber $\mathcal{X}_\kappa$ a certain "overconvergent space $\mathcal{W}$ over K." The de Rham cohomology of this space computes the Monsky–Washnitzer cohomology of $\mathcal{X}_\kappa$. We may further equip $\mathcal{W}$ with a K-morphism ϕ lifting the relative Frobenius morphism of $\mathcal{X}_\kappa/\kappa$. Under these assumptions, the finite polynomial cohomology associated with a fixed polynomial $P(t)$ is

$$H^i_{\mathrm{f},P}(\mathcal{X}, n) = H^i(\mathrm{MF}(F^n\Omega^\bullet_{\log}(X) \xrightarrow{P(\phi^*)} \Omega^\bullet(\mathcal{W}))),$$

where $\Omega^\bullet_{\log}(X)$ is the complex of differential forms on X with logarithmic singularities at infinity (relative to some compactification) and the map corresponds to restricting differential forms to $\mathcal{W}$ and then applying the map $P(\phi^*)$. The restriction map only exists in the derived category so this is somewhat imprecise but we do not say much more about this as we will not be using this directly. The notation MF refers to the mapping fiber of a map, which is the cone shifted by -1. To get the finite polynomial cohomology $H^i_{\mathrm{fp},R}(\mathcal{X}, n)$, one takes the limit of the above groups over all polynomials P whose roots have weights in R relative to some transition maps defined in [6, Definition 2.3].

For the actual computations, we will use a modified version, where algebraic forms are replaced by rigid forms,

$$\tilde{H}^i_{\mathrm{f},P}(\mathcal{X}, n) = H^i(\mathrm{MF}(F^n\Omega^\bullet(\mathcal{W}) \xrightarrow{P(\phi^*)} \Omega^\bullet(\mathcal{W}))),$$

where $F^n\Omega^\bullet(\mathcal{W})$ is just the stupid filtration. We again take the limit over all P of the given weights R to obtain $\tilde{H}^i_{\mathrm{fp},R}(\mathcal{X}, n)$. Because the modified finite polynomial cohomology does not use forms with log singularities, there would be situations where it is infinite dimensional for open schemes, and it will certainly differ from the non-modified version. However, when our ultimate goal is a computation in a proper scheme, or in other situations when the cohomologies do not differ (see below), it can be used for these computations and in some cases, as we will see, offers some benefits over the original.

An element of $\tilde{H}^i_{f,P}(\mathcal{X}, n)$ is given explicitly by a pair

$$\tilde{\omega} = (\omega, f), \quad \omega \in F^n\Omega^i(\mathcal{W}) \text{ closed}, \quad f \in \Omega^{i-1}(\mathcal{W}), \quad df = P(\phi^*)\omega. \tag{2.6}$$

Pairs of the form $(dg, P(\phi^*)g)$ with $g \in F^n\Omega^{i-1}(\mathcal{W})$ and $(0, dh)$ with $h \in \Omega^{i-2}(\mathcal{W})$ are identified with 0.

We have obvious maps $H^i_{f,P}(\mathcal{X}, n) \to \tilde{H}^i_{f,P}(\mathcal{X}, n)$, and, taking the limit over P, $H^i_{\mathrm{fp},R}(\mathcal{X}, n) \to \tilde{H}^i_{\mathrm{fp},R}(\mathcal{X}, n)$, hence also $H^i_{\mathrm{fp}}(\mathcal{X}, n) \to \tilde{H}^i_{\mathrm{fp}}(\mathcal{X}, n)$ (Recalling that this means $R = \{2i\}$). These are compatible with cup products. Considering now H^2 for our affine $\mathcal{X}$ of relative dimension 1 we find, as $F^2 = 0$ on both H^1_{rig} and H^1_{dR},

$$H^2_{\mathrm{fp}}(\mathcal{X}, 2) = \tilde{H}^2_{\mathrm{fp}}(\mathcal{X}, 2) = H^1_{\mathrm{rig}}(\mathcal{X}_\kappa/K) = H^1_{\mathrm{dR}}(\mathcal{W})$$

Thus, to compute the cup product

$$H^1_{\mathrm{fp}}(\mathcal{X}, 1) \times H^1_{\mathrm{fp}}(\mathcal{X}, 1) \to H^2_{\mathrm{fp}}(\mathcal{X}, 2),$$

we may pass to $\tilde{H}^1_{\mathrm{fp}}(\mathcal{X}, 1)$ and compute the cup product there.

It is important to note that the natural isomorphism $\boldsymbol{i} : H^1_{\mathrm{dR}}(\mathcal{W}) \to \tilde{H}^2_{\mathrm{fp}}(\mathcal{X}, 2)$ is not the obvious one but is rather given by the limit of the isomorphisms, which we abuse notation to denote again by $\boldsymbol{i}$,

$$H^1_{\mathrm{dR}}(\mathcal{W}) \xrightarrow{\boldsymbol{i}} \tilde{H}^2_{\mathrm{fp},P}(\mathcal{X}, 2), \quad [\omega] \mapsto P(\phi^*)[\omega], \tag{2.7}$$

where [ ] denoted the cohomology class of a form. This is necessary because of the transitions maps between different polynomials.

The cup product in fp-cohomology is described in [6, Remark 4.3]. Specializing the construction there to our case we find it is given as the limit of unnormalized cup products

$$\tilde{\cup} : \tilde{H}^1_{f,P}(\mathcal{X}, 1) \times \tilde{H}^1_{f,Q}(\mathcal{X}, 1) \to \tilde{H}^2_{f,P*Q}(\mathcal{X}, 2) = H^1_{\mathrm{dR}}(\mathcal{W}),$$

where by definition $\prod(1 - \alpha_i t) * \prod(1 - \beta_j t) = \prod(1 - \alpha_i\beta_j t)$, then twisted back by (2.7). The following proposition describes the cup product in the case that we will require.

Proposition 2.2 *For* $i = 1, 2$ *let* $P_i(t) = (1 - \alpha_i t)$ *and suppose that* $\tilde{\omega}_i = (\omega_i, G_i^{(p)}) \in \tilde{H}^1_{f,P_i}(\mathcal{X}, 1)$ *so that* $(1 - \alpha_i\phi^*)\omega_i = dG_i^{(p)}$. *Then,*

$$\tilde{\omega}_1 \tilde{\cup} \tilde{\omega}_2 = (0, G_1^{(p)}\omega_2 - \alpha_1 G_2^{(p)}\phi^*\omega_1) \in \tilde{H}^2_{f,P_1*P_2}(\mathcal{X}, 2).$$

In other words, it is identified with the cohomology class of the form $G_1^{(p)}\omega_2 - \alpha_1 G_2^{(p)}\phi^*\omega_1$ *in* $H^1_{dR}(\mathcal{W})$ *in the non-normalized identification.*

Proof This follows directly from the description of the cup product in [6, Remark 4.3] and from the fact that in the polynomial ring in two variables $K[t_1, t_2]$ we have

$$1 - \alpha_1\alpha_2 t_1 t_2 = (1 - \alpha_1)t_1 + \alpha_1 t_1(1 - \alpha_2 t_2).$$

□

By [6] (see also [9] Definition 2.13, Remark 2.14 and the preceding discussion), classes in finite polynomial cohomology can be interpreted as Coleman functions. In particular, the class $\tilde{\omega} = (\omega, g) \in \tilde{H}^1_{f,P}(\mathcal{X}, 1)$ corresponds to the primitive F_ω of ω (a locally analytic function with $dF_\omega = \omega$) satisfying the relation $P(\phi^*)F_\omega = g$. The value $F_\omega(x)$ at an $\mathcal{O}_K$-section x is just the pullback of $\tilde{\omega}$ to $\operatorname{Spec}(\mathcal{O}_K)$ via x,

$$F_\omega(x) = x^*\tilde{\omega}, \tag{2.8}$$

via a natural identification of $\tilde{H}^1_{f,P}(\operatorname{Spec}(\mathcal{O}_K), 1)$ with K. We also note that

$$\tilde{H}^1_{f,P}(\operatorname{Spec}(\mathcal{O}_K), 0) = 0 \tag{2.9}$$

since $H^0_{\mathrm{dR}}(K/K)/F^0 = 0$.

We next discuss yet another simple modification of fp-cohomology (this is a new construction).

Definition 2.3 Suppose K' is a finite extension of K and let $\mathcal{X}' := \mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$. We define fp-cohomology of $\mathcal{X}'$ over K to be the cohomology of the fp-complex of $\mathcal{X}$ but tensored over K with K'. This construction applies to all versions of fp-cohomology. We will denote this cohomology by $H^i_{\mathrm{fp}}(\mathcal{X}', n)_K$.

Because de Rham and rigid complexes behave well under finite base change, this construction is equivalent to writing down the fp-complex of $\mathcal{X}'$ but using the polynomials and the lift of Frobenius for $\mathcal{X}$ based changed to $\mathcal{X}'$, instead of the polynomials and the lift of Frobenius on $\mathcal{X}'$. One easily sees that we are in effect taking the limit over a larger class of morphisms, but with the old class being cofinal in them. We therefore have a natural isomorphism

$$H^i_{\mathrm{fp}}(\mathcal{X}', n)_K \cong H^i_{\mathrm{fp}}(\mathcal{X}', n). \tag{2.10}$$

All of what we said about fp-cohomology extends immediately to this setting.

Suppose now that $\mathcal{X}$ is the localization of a smooth complete $\mathcal{O}_K$-scheme $\mathcal{C}$ of relative dimension 1 with generic fiber C at a finite number of sections. There is then a unique Frobenius equivariant splitting Ψ to the restriction map in de Rham cohomology $H^1_{\mathrm{dR}}(C/K) \to H^1_{\mathrm{dR}}(\mathcal{W}/K)$ [8, Proposition 4.8]. Composing with the cup product pairing on $H^1_{\mathrm{dR}}(C/K)$, we obtain a pairing

$$\langle \bullet, \bullet \rangle : H^1_{\mathrm{dR}}(C/K) \times H^1_{\mathrm{dR}}(\mathcal{W}/K) \to K, \tag{2.11}$$

which is Frobenius equivariant in the sense that

$$\left\langle \phi^*\eta, \phi^*\omega \right\rangle = p\langle \eta, \omega\rangle \tag{2.12}$$

and compatible with the cup product in the sense that

$$\langle \eta, \omega|_{\mathcal{W}}\rangle = \operatorname{tr}(\eta \cup_{\mathrm{dR}} \omega), \quad \omega, \eta \in H^1_{\mathrm{dR}}(C/K). \tag{2.13}$$

The pairing above is also compatible with restriction in the sense that for a smaller overconvergent space $\mathcal{W}'$ we have

$$\langle \eta, \omega|_{\mathcal{W}'}\rangle_{\mathcal{W}'} = \langle \eta, \omega\rangle_{\mathcal{W}}. \tag{2.14}$$

3 Modular forms

Keeping the notations of the previous section, suppose now that N is an integer prime to p. Let K be some finite extension of $\mathbb{Q}_p$. We let $\mathcal{X}_1(N)$ be a model for the modular curve $X_1(N)$ over $\mathbb{Z}_p$, base changed to $\mathcal{O}_K$, and let $\mathcal{X}$ be the affine scheme obtained by removing from $\mathcal{X}_1(N)$ sections at lifting the supersingular points (we want to do it carefully, to get something base changed from $\mathbb{Z}_p$), and $\mathcal{X}'$ the one obtained by further removing the sections at the cusps.

When doing computations with the fp-cohomology of $\mathcal{X}$, we can use the following relations with the theory of (p-adic) modular forms. The space $\mathcal{W}$ is obtained by throwing away from $X_1(N)$ "infinitesimally less" then the supersingular disks (the space $\mathcal{W}'$ corresponding to $\mathcal{X}'$ is obtained by further throwing away disks around the cusps). The ring $\mathcal{O}(\mathcal{W})$ (respectively, the module of differentials $\Omega^1(\mathcal{W})$) is naturally identified with the space $M_0(N)$ (respectively, $S_2(N)$) of overconvergent p-adic modular forms (respectively, cusp forms) of weight 0 (respectively, 2). In terms of q-expansions, weight 0 overconvergent modular forms are by definition functions on $\mathcal{W}$, while the weight 2 form with expansion $f(q)$ is identified with the differential one form ω_f with expansion

$$\omega_f = f(q)\mathrm{d}q/q. \tag{3.1}$$

In particular, one finds that differentiation translates to the θ operator on p-adic modular forms given on q-expansions by

$$\theta\left(\sum a_n q^n\right) = \sum n a_n q^n \tag{3.2}$$

so that

$$df = \omega_{\theta(f)},\ f \in M_0(N). \tag{3.3}$$

The space $\mathcal{W}$ carries the canonical lift of Frobenius ϕ defined in terms of moduli by quotienting out by the canonical subgroup (note that this would be the base change of the linear Frobenius over $\mathbb{Q}_p$, so is appropriate for doing fp-cohomology "over $\mathbb{Q}_p$" as in Definition 2.3).

Its action on the parameter q at ∞ is $\phi^* q = q^p$. Its action on one form is therefore given by

$$\phi^*\omega_f = p\omega_{Vf}, \tag{3.4}$$

where the operator V on p-adic modular forms is given on q-expansions by

$$V\left(\sum a_n q^n\right) = \sum a_n q^{pn}. \tag{3.5}$$

The U operator on p-adic modular forms is given on q-expansions by $U(\sum a_n q^n) = \sum a_{np} q^n$. We clearly have $UV = \mathrm{Id}$ and

$$g - VUg = g^{[p]}, \tag{3.6}$$

where $g^{[p]}$ is the *p-depletion* of g, which has, if g has q-expansion $\sum a_n q^n$, a q-expansion $\sum_{p\nmid n} a_n q^n$. It is a fact that there exists an overconvergent modular form $\theta^{-1}g^{[p]}$ with

$$g^{[p]} = \theta(\theta^{-1}g^{[p]}). \tag{3.7}$$

We normalize $\theta^{-1}g^{[p]}$ to have expansion $\sum_{p\nmid n} \frac{a_n}{n} q^n$. In particular, note that

$$U\theta^{-1}g^{[p]} = 0. \tag{3.8}$$

We remark that Bertolini, Darmon and Rotger denote $\theta^{-1}g^{[p]}$ by $d^{-1}g^{[p]}$.

The relation between the U and V operators, and the Hecke operator T_p on the space $S_2(N, \chi)$ of weight 2 cusp forms on $\Gamma_1(N)$ with Nebentypus character χ is given as follows [14, 8.2.2],

$$T_p = U + \chi(p)p^{k-1}V. \tag{3.9}$$

We will need the following result (this is given in [11] subsection 2.6).

Proposition 3.1 *Suppose that $g \in S_2(N, \chi)$ is an eigenvector for T_p with eigenvalue a_p. Then*

$$\omega_g - a_p p^{-1}\phi^*\omega_g + p^{-1}\chi(p)(\phi^*)^2\omega_g = \omega_{g^{[p]}} = \mathrm{d}\theta^{-1}g^{[p]}.$$

Proof By (3.9), we have

$$a_p g = T_p g = Ug + \chi(p)pVg.$$

Multiplying by V we have

$$a_p Vg = VUg + \chi(p)pV^2 g.$$

Adding $g - VUg - a_p Vg$ to both sides gives, using (3.6)

$$g^{[p]} = g - VUg = g - a_p Vg + \chi(p)pV^2 g.$$

Rewriting in terms of pV gives

$$g^{[p]} = g - a_p p^{-1}(pV)g + \chi(p)p^{-1}(pV)^2 g.$$

Using (3.3) and (3.4) gives the result. □

Hida defined the ordinary projection

$$e_{\mathrm{ord}} = \lim U^{n!} \tag{3.10}$$

on the space of weight k overconvergent modular forms. Its image is the space of ordinary forms. We will in particular import its action to the space $\Omega^1(\mathcal{W})$ via the identification above. We list the following properties of e_{ord} which may be found in [11].

Proposition 3.2 *The projection e_{ord} satisfies the following properties.*

(1) *It vanishes on* $G\phi^*\omega$ *whenever* $G \in \mathcal{O}(\mathcal{W})$ *and* $UG = 0$.
(2) *If* $\eta \in H^1_{dR}(X_1(N)/K)$ *is in the unit root part (i.e., an eigenvector for* ϕ *whose eigenvalue is a p-adic unit), then* $\langle \eta, \omega\rangle = \langle \eta, e_{\mathrm{ord}}\omega\rangle$.

Proof For (1), we notice that the conditions imply that in G the coefficients of q^{np} are 0 for all n. On the other hand, by (3.4) and (3.5), in the q-expansion of $\phi^*\omega$, $a_n \neq 0$ only if $p|n$. This implies that in the q-expansion of $G\phi^*\omega$ the coefficients of q^{np} are 0 for all n and so $U(G\phi^*\omega) = 0$, from which it clearly follows that $e_{\mathrm{ord}}(G\phi^*\omega) = 0$ by (3.10). For (2), we observe that U and V are inverses of each other on $H^1_{\mathrm{dR}}(\mathcal{W})$ and as $\phi^* = pV$ we have

$$\langle \eta, U\omega\rangle = \left\langle \eta, p(\phi^*)^{-1}[\omega]\right\rangle = p^{-1}\left\langle \phi^*\eta, p\omega\right\rangle = \left\langle \phi^*\eta, \omega\right\rangle.$$

Thus, if $\phi^*\eta = \beta\eta$ we have

$$\left\langle \eta, U^n\omega\right\rangle = \beta^n \langle \eta, \omega\rangle$$

and if $|\beta| = 1$, taking the limit (and noting that $\langle \bullet, \bullet\rangle$ is continuous, as can be observed by writing it in terms of residues as in [8, Proposition 4.10]), we get

$$\langle \eta, e_{\mathrm{ord}}\omega\rangle = \lim_n (\beta^{n!})\langle \eta, \omega\rangle = \langle \eta, \omega\rangle$$

as required. □

Theorem 3.3 *Suppose that for* $\omega_1, \omega_2 \in \Omega^1(\mathcal{W})$ *and* $G_1, G_2 \in \mathcal{O}(\mathcal{W})$ *we have* $\omega_i - \alpha_i\phi^*\omega_i = dG_i$ *for* $i = 1, 2$ *with* $UG_i = 0$ *and let* $\tilde{\omega}_i = (\omega_i, G_i) \in \tilde{H}^1_{\mathrm{f},1-\alpha_i t}(\mathcal{X}, 1)_{\mathbb{Q}_p}$ *as in* (2.6), *in the version of Definition* 2.3. *Let* $P(t) = 1-\alpha_1\alpha_2 t$. *Consider* $\tilde{\omega}_1 \cup \tilde{\omega}_2 \in \tilde{H}^2_{\mathrm{fp},P}(\mathcal{X}, 2)_{\mathbb{Q}_p}$ *and its image* $\boldsymbol{i}^{-1}(\tilde{\omega}_1 \cup \tilde{\omega}_2) \in H^1_{dR}(\mathcal{W})$ *under the inverse of the normalized identification* (2.7). *If* $\eta \in H^1_{dR}(X_1(N)/K)$ *is an eigenvector for* ϕ^* *with eigenvalue* α *and* $|\alpha| = 1$, *then*

$$\left\langle \eta, \boldsymbol{i}^{-1}(\tilde{\omega}_1 \cup \tilde{\omega}_2)\right\rangle = (1 - \beta\alpha_1\alpha_2)^{-1}\left\langle \eta, e_{\mathrm{ord}}(G_1\omega_2)\right\rangle$$

with $\beta = p/\alpha$. *The conditions may be relaxed to allow (one, or both)* ω_i *to have logarithmic poles at the cusps if* $\alpha_i = 1/p$ *when considering* $\tilde{\omega}_1 \cup \tilde{\omega}_2 \in \tilde{H}^2_{\mathrm{fp},P}(\mathcal{X}', 2)_{\mathbb{Q}_p} \cong H^1_{dR}(\mathcal{W}')$.

Proof By Proposition 2.2, we have

$$\tilde{\omega}_1 \tilde{\cup} \tilde{\omega}_2 = (0, G_1\omega_2 - \alpha_1 G_2\phi^*\omega_1).$$

We have, by the assumption that $UG_2 = 0$ and by (1) of Proposition 3.2

$$e_{\mathrm{ord}}(G_2\phi^*\omega_1) = 0.$$

Since η is in the unit root subspace, it follows from (2) of Proposition 3.2 that, identifying $\tilde{\omega}_1 \tilde{\cup} \tilde{\omega}_2$ with its image in $H^1_{\mathrm{dR}}(\mathcal{W})$,

$$\left\langle \eta, \tilde{\omega}_1 \tilde{\cup} \tilde{\omega}_2\right\rangle = \left\langle \eta, e_{\mathrm{ord}}(G_1\omega_2 - \alpha_1 G_2\phi^*\omega_1)\right\rangle = \left\langle \eta, e_{\mathrm{ord}}(G_1\omega_2)\right\rangle.$$

The normalization (2.7) means that the cohomology class $\boldsymbol{i}^{-1}(\tilde{\omega}_1 \cup \tilde{\omega}_2)$ satisfies $(1 - \alpha_1\alpha_2\phi^*)\boldsymbol{i}^{-1}(\tilde{\omega}_1 \cup \tilde{\omega}_2) = \tilde{\omega}_1 \tilde{\cup} \tilde{\omega}_2$. Pairing with η and using the property (2.12) of the pairing, we obtain

$$\begin{aligned}\langle \eta, e_{\mathrm{ord}}(G_2\omega_1)\rangle &= \langle \eta, \tilde{\omega}_1 \tilde{\cup} \tilde{\omega}_2\rangle \\ &= \langle \eta, (1-\alpha_1\alpha_2\phi^*)\boldsymbol{i}^{-1}(\tilde{\omega}_1 \cup \tilde{\omega}_2)\rangle \\ &= \langle \eta, \boldsymbol{i}^{-1}(\tilde{\omega}_1 \cup \tilde{\omega}_2)\rangle) - \alpha_1\alpha_2\langle \alpha^{-1}\phi^*\eta, \phi^*\boldsymbol{i}^{-1}(\tilde{\omega}_1 \cup \tilde{\omega}_2)\rangle \\ &= \left(1 - \frac{p\alpha_1\alpha_2}{\alpha}\right)\langle \eta, \boldsymbol{i}^{-1}(\tilde{\omega}_1 \cup \tilde{\omega}_2)\rangle,\end{aligned}$$

which gives the result. To see that we may allow logarithmic poles at the cusps when $\alpha_i = 1/p$, note that $\omega_i - \phi^*\omega_i/p$ is holomorphic at the cusps. Thus, the same argument can be carried through. □

As a corollary, we give a formula for a similar product where the forms are associated with weight two eigenforms. For the case where the forms are Eisenstein series, we will need to recall the notion of a constant term of a Coleman function [3, Definition 7.7]. When a form ω has a logarithmic singularity at x with residue a, its Coleman integral is given, with respect to a local parameter z at x, by $F_\omega = a\log(z) + \sum_{n=0}^{\infty} a_n z^n$. The constant term of F_ω with respect to the parameter z is a_0.

Corollary 3.4 *Let $g = \sum a_n(g)q^n$ and $h = \sum a_n(h)q^n$ be cusp forms for $\Gamma_1(N)$, with characters χ_g and χ_h, respectively, which are eigenforms for the p-th Hecke operator T_p and let ω_g and ω_h be the associated one forms on $X_1(N)$. Let $\tilde{\omega}_g, \tilde{\omega}_h \in H^1_{\mathrm{fp}}(\mathcal{X}_1(N), 1)$ be lifts of ω_g and ω_h, respectively, as in (2.2), such that the associated Coleman functions vanish at ∞. Let $\alpha_g, \alpha_h, \beta_g, \beta_h$ be determined (up to switching the α's by the β's) by the formulas*

$$\begin{aligned}1 - a_p(g)p^{-1}t + p^{-1}\chi_g(p)t^2 &= (1-\alpha_g t)(1-\beta_g t),\\ 1 - a_p(h)p^{-1}t + p^{-1}\chi_h(p)t^2 &= (1-\alpha_h t)(1-\beta_h t).\end{aligned}$$

Let $\eta \in H^1_{dR}(X_1(N)/K)$ be an eigenvector for ϕ^* with eigenvalue α and $|\alpha| = 1$. Let $\beta = p/\alpha$. Then we have

$$\begin{aligned}\langle \eta, \boldsymbol{i}^{-1}(\tilde{\omega}_g \cup \tilde{\omega}_h)|_{\mathcal{W}}\rangle = &(1-\beta^2\alpha_g\alpha_h\beta_g\beta_h)\\ &\times \left((1-\beta\alpha_g\alpha_h)(1-\beta\beta_g\alpha_h)(1-\beta\alpha_g\beta_h)(1-\beta\beta_g\beta_h)\right)^{-1}\\ &\times \left\langle \eta, e_{\mathrm{ord}}(\theta^{-1}g^{[p]}\omega_h)\right\rangle,\end{aligned}$$

where $\theta^{-1}g^{[p]}$ is defined in (3.7). The same conclusion holds when g or h are replaced by Eisenstein series, provided that $\mathcal{W}$ is replaced by $\mathcal{W}'$ and the value at ∞ is taken in the sense of the constant term with respect to the standard parameter q.

Proof By functoriality, we may compute $\boldsymbol{i}^{-1}(\tilde{\omega}_g \cup \tilde{\omega}_h)|_{\mathcal{W}}$ by first restricting $\tilde{\omega}_g$ and $\tilde{\omega}_h$ to $\mathcal{X}$, taking their cup products and applying $\boldsymbol{i}^{-1}$, and we may use the explicit formulas developed in this section and in particular in Theorem 3.3, using the version "over $\mathbb{Q}_p$" of Definition 2.3 and the canonical Frobenius lift ϕ. Set

$$P(t) = (1-\alpha_g t)(1-\beta_g t) = 1 - a_p(g)p^{-1}t + p^{-1}\chi_g(p)t^2\,.$$

By Proposition 3.1, we have the relation $P(\phi^*)\omega_g = d\theta^{-1}g^{[p]}$, from which we obtain a class $(\omega_g, \theta^{-1}g^{[p]}) \in \tilde{H}^1_{f,P}(\mathcal{X}, 1)_{\mathbb{Q}_p} \subset \tilde{H}^1_{\mathrm{fp}}(\mathcal{X}, 1)_{\mathbb{Q}_p}$. Since $\theta^{-1}g^{[p]}$ vanishes at infinity, so does the associated Coleman function F_g (it satisfies the relation $P(\phi^*)F_g = \theta^{-1}g^{[p]}$ and ϕ fixes o). Consequently, this is the restriction of $\tilde{\omega}_g$ to $\mathcal{X}$.

We let

$$\begin{aligned} \omega_{g,\alpha} &= (1-\beta_g\phi^*)\omega_g, \quad \omega_{g,\beta} = (1-\alpha_g\phi^*)\omega_g, \\ \omega_{h,\alpha} &= (1-\beta_h\phi^*)\omega_h, \quad \omega_{h,\beta} = (1-\alpha_h\phi^*)\omega_h. \end{aligned} \tag{3.11}$$

These may be lifted to classes

$$\begin{aligned} \tilde{\omega}_{g,\alpha} &= (\omega_{g,\alpha}, \theta^{-1}g^{[p]}) \in \tilde{H}^1_{f,1-\alpha_g t}(\mathcal{X}, 1)_{\mathbb{Q}_p} \subset \tilde{H}^1_{\mathrm{fp}}(\mathcal{X}, 1)_{\mathbb{Q}_p}, \\ \tilde{\omega}_{g,\beta} &= (\omega_{g,\beta}, \theta^{-1}g^{[p]}) \in \tilde{H}^1_{f,1-\beta_g t}(\mathcal{X}, 1)_{\mathbb{Q}_p} \subset \tilde{H}^1_{\mathrm{fp}}(\mathcal{X}, 1)_{\mathbb{Q}_p}, \end{aligned}$$

and similarly for h in place of g. As Coleman functions, they vanish at ∞ for the same reason that F_g vanishes there. We clearly have $\omega_g = (\alpha_g - \beta_g)^{-1}(\alpha_g\omega_{g,\alpha} - \beta_g\omega_{g,\beta})$ and the vanishing at infinity immediately implies the same relation on Coleman functions, hence

$$(\tilde{\omega}_g)|_{\mathcal{X}} = (\alpha_g - \beta_g)^{-1}(\alpha_g\tilde{\omega}_{g,\alpha} - \beta_g\tilde{\omega}_{g,\beta}). \tag{3.12}$$

We have a similar decomposition with g replaced by h. It remains to compute the right-hand side of (4.3) using the bilinearity of the cup product, Theorem 3.3 and the decomposition (3.12) and its h-analog. There will be 4 summands (and a common multiple of $((\alpha_g - \beta_g)(\alpha_h - \beta_h))^{-1}$), and using the relation

$$e_{\mathrm{ord}}(\theta^{-1}g^{[p]}\omega_{h,\alpha}) = e_{\mathrm{ord}}(\theta^{-1}g^{[p]}\omega_h) = e_{\mathrm{ord}}(\theta^{-1}g^{[p]}\omega_{h,\beta})$$

coming from (3.11) and 1 of Proposition 3.2, they will all be constant multiples of $\langle \eta, e_{\mathrm{ord}}(\theta^{-1}g^{[p]}\omega_h)\rangle$. For example, the first term would be

$$\begin{aligned} \alpha_g\alpha_h\langle \eta, \boldsymbol{i}^{-1}(\tilde{\omega}_{g,\alpha} \cup \tilde{\omega}_{h,\alpha})\rangle &= \alpha_g\alpha_h(1-\beta\alpha_g\alpha_h)^{-1}\Big\langle \eta, e_{\mathrm{ord}}(\theta^{-1}g^{[p]}\omega_{h,\alpha})\Big\rangle \\ &= \alpha_g\alpha_h(1-\beta\alpha_g\alpha_h)^{-1}\Big\langle \eta, e_{\mathrm{ord}}(\theta^{-1}g^{[p]}\omega_h).\Big\rangle \end{aligned}$$

One is then left with a slightly tedious computation of the sum of the constant multiples giving

$$\begin{aligned} &((\alpha_g - \beta_g)(\alpha_h - \beta_h))^{-1}\left(\frac{\alpha_g\alpha_h}{1-\beta\alpha_g\alpha_h} - \frac{\beta_g\alpha_h}{1-\beta\beta_g\alpha_h} - \frac{\alpha_g\beta_h}{1-\beta\alpha_g\beta_h} + \frac{\beta_g\beta_h}{1-\beta\beta_g\beta_h}\right) \\ &\quad = (1-\beta^2\alpha_g\alpha_h\beta_g\beta_h)\left((1-\beta\alpha_g\alpha_h)(1-\beta\beta_g\alpha_h)(1-\beta\alpha_g\beta_h)(1-\beta\beta_g\beta_h)\right)^{-1}. \end{aligned}$$

Finally, the argument extends without any difficulty to the case of Eisenstein series. The only point to note is $\phi^* - p$ kills $\log(q)$, so that having constant term 0 for F_{ω_g} with respect to q implies that $P(\phi^*)F_g$ vanishes at ∞ as before. □

4 Diagonal cycles

In this section, we study the diagonal, or Gross-Kudla-Schoen, cycle, introduced in [12] and [13] and studied p-adically in [11]. It is a homologically trivial cycle on the triple product $\mathcal{X}_1(N)^3$. To define it, we first pick a base section o.

Definition 4.1 Define, for a non-empty subset $I \subset \{1,2,3\}$, the partial diagonal $\Delta_I : \mathcal{X}_1(N) \to \mathcal{X}_1(N)^3$ by the formula

$$(\Delta_I(x))_i = \begin{cases} x & i \in I \\ o & \text{otherwise.} \end{cases}$$

Then, abusing the notation to write Δ_I for the cycle $(\Delta_I)_* \mathcal{X}_1(N)$, we define the cycle Δ by

$$\begin{aligned} \Delta &= \Delta_{\{1,2,3\}} - \Delta_{\{1,2\}} - \Delta_{\{1,3\}} - \Delta_{\{2,3\}} + \Delta_{\{1\}} + \Delta_{\{2\}} + \Delta_{\{3\}} \\ &= - \sum_{\emptyset \neq I \subset \{1,2,3\}} (-1)^{|I|} \Delta_I \in \mathrm{CH}^2(\mathcal{X}_1(N)^3). \end{aligned}$$

Since Δ is homologically trivial, one can apply the p-adic Abel-Jacobi map [6] to it and obtain

$$\mathrm{AJ}_p(\Delta) \in H^3_{\mathrm{dR}}(X_1(N)^3/\mathbb{Q}_p)/F^2 = \mathrm{Hom}(F^2 H^3_{\mathrm{dR}}(X_1(N)^3/\mathbb{Q}_p), \mathbb{Q}_p). \tag{4.1}$$

Let K be a sufficiently large finite extension of $\mathbb{Q}_p$. This should include the coefficients of the forms in the theorem to follow. Our goal is to deduce the following result of Darmon and Rotger, essentially [11, Theorem 3.14].

Theorem 4.2 *Suppose o is the cusp at infinity. Let $g = \sum a_n(g) q^n$ and $h = \sum a_n(h) q^n$ be cusp forms for $\Gamma_1(N)$, with characters χ_g and χ_h, respectively, which are eigenforms for the p-th Hecke operator T_p and let ω_g and ω_h be the associated one forms on $X_1(N)$. Let $\alpha_g, \alpha_h, \beta_g, \beta_h$ be determined (up to switching the α's by the β's) by the formulas*

$$\begin{aligned} 1 - a_p(g) p^{-1} t + p^{-1} \chi_g(p) t^2 &= (1 - \alpha_g t)(1 - \beta_g t), \\ 1 - a_p(h) p^{-1} t + p^{-1} \chi_h(p) t^2 &= (1 - \alpha_h t)(1 - \beta_h t). \end{aligned}$$

Let $\eta \in H^1_{dR}(X_1(N)/K)$ be an eigenvector for ϕ^* with eigenvalue α and $|\alpha| = 1$. Let $\beta = p/\alpha$. Then we have

$$\begin{aligned} \mathrm{AJ}_p(\Delta)(\eta \cup \omega_g \cup \omega_h) = (1 - \beta^2 \alpha_g \alpha_h \beta_g \beta_h) & \\ \times \left((1 - \beta\alpha_g\alpha_h)(1 - \beta\beta_g\alpha_h)(1 - \beta\alpha_g\beta_h)(1 - \beta\beta_g\beta_h)\right)^{-1} & \\ \times \left\langle \eta, e_{\mathrm{ord}}(\theta^{-1} g^{[p]} \omega_h) \right\rangle, & \end{aligned}$$

where $\theta^{-1} g^{[p]}$ is defined in (3.7).

Proof Let $\pi_i : \mathcal{X}_1(N)^3 \to \mathcal{X}_1(N)$ denote the projection on the i-th component. For any lifts $\tilde{\omega}_g, \tilde{\omega}_h \in H^1_{\mathrm{fp}}(\mathcal{X}_1(N), 1)$ of ω_g and ω_h, respectively, as in (2.2), and for η, viewed as an element of $H^1_{\text{fp}}(\mathcal{X}_1(N), 0)$ as in (2.3), The class $\pi_1^*\eta \cup \pi_2^*\tilde{\omega}_g \cup \pi_3^*\tilde{\omega}_h$ is a lift of $\eta \cup \omega_g \cup \omega_h$ to $H^3_{\text{fp}}(\mathcal{X}_1(N)^3, 2)$. By [6, Theorem 1.2], we have

$$\mathrm{AJ}_p(\Delta)(\eta \cup \omega_g \cup \omega_h) = -\sum_{\emptyset \neq I \subset \{1,2,3\}} (-1)^{|I|} \operatorname{tr} \Delta_I^*(\pi_1^*\eta \cup \pi_2^*\tilde{\omega}_g \cup \pi_3^*\tilde{\omega}_h),$$

where $\operatorname{tr} : H^3_{\text{fp}}(\mathcal{X}_1(N), 2) \to K$ is the trace in fp-cohomology, $H^3_{\text{fp}}(\mathcal{X}_1(N), 2) \xrightarrow{\boldsymbol{i}^{-1}} H^2_{\text{dR}}(X_1(N)/K) \xrightarrow{\text{tr}} K$ (see Proposition 2.1).

Recall from Sect. 2 that fp-cohomology classes correspond to Coleman functions in such a way that the value of these Coleman functions at a point is obtained by pullback. Let F_g and F_h be the Coleman functions associated with $\tilde{\omega}_g$ and $\tilde{\omega}_h$, respectively. Since $\pi_i \circ \Delta_I$ is the identity if $i \in I$ and the map o sending every x to o otherwise, we see that $\mathrm{AJ}_p(\Delta)(\eta \cup \omega_g \cup \omega_h)$ is the trace of

$$\begin{aligned}
&\eta \cup \tilde{\omega}_g \cup \tilde{\omega}_h \\
&\quad - \eta \cup \tilde{\omega}_g \cup o^*\tilde{\omega}_h - \eta \cup o^*\tilde{\omega}_g \cup \tilde{\omega}_h \\
&\quad - o^*\eta \cup \tilde{\omega}_g \cup \tilde{\omega}_h \\
&\quad + \eta \cup o^*(\tilde{\omega}_g \cup \tilde{\omega}_h) \\
&\quad + o^*\eta \cup \tilde{\omega}_g \cup o^*\tilde{\omega}_h \quad + o^*\eta \cup o^*\tilde{\omega}_g \cup \tilde{\omega}_h.
\end{aligned}$$

In this expression, the third and fifth lines are 0 since $o^*\eta \in \tilde{H}^1_{f,P}(\operatorname{Spec}(\mathcal{O}_K), 0)$, which is 0 by (2.9), and the fourth line is 0 since $\mathcal{O}_K$ has no second fp-cohomology. From this and (1) of Proposition 2.1, it is now clear that

$$\begin{aligned}\mathrm{AJ}_p(\Delta)(\eta \cup \omega_g \cup \omega_h) = \operatorname{tr}\big(&\eta \cup_{\text{dR}} \boldsymbol{i}^{-1}(\tilde{\omega}_g \cup \tilde{\omega}_h) \\ &\pm F_g(o)\eta \cup_{\text{dR}} \omega_h \pm F_h(o)\eta \cup_{\text{dR}} \omega_g\big).\end{aligned} \tag{4.2}$$

We will pick $\tilde{\omega}_g$ and $\tilde{\omega}_h$ in such a way that the associated F_g and F_h vanish at o. Thus we get

$$\mathrm{AJ}_p(\Delta)(\eta \cup \omega_g \cup \omega_h) = \eta \cup_{\text{dR}} \boldsymbol{i}^{-1}(\tilde{\omega}_g \cup \tilde{\omega}_h).$$

By (2.13) we thus find

$$\mathrm{AJ}_p(\Delta)(\eta \cup \omega_g \cup \omega_h) = \langle \eta, \boldsymbol{i}^{-1}(\tilde{\omega}_g \cup \tilde{\omega}_h)|_{\mathcal{W}} \rangle \tag{4.3}$$

and the result follows immediately from Corollary 3.4. □

Remark 4.3 From (4.2) one can see that the choice of o is essential. Changing o to o' changes the left-hand side of (4.1) by $\pm \int_o^{o'} \omega_g \eta \cup_{\text{dR}} \omega_h \pm \int_o^{o'} \omega_h \eta \cup_{\text{dR}} \omega_g$ and this does not seem to be 0 in general. In [11], the assumption that o is the cusp is implicit in formula (95), as it is easy to see that this only holds under this assumption.

Remark 4.4 To relate our result with that of [11], we note that our reciprocal roots are related to the ones used there by

$$\alpha_g = p^{-1}\alpha_g^{\text{DaRo}}, \quad \beta_g = p^{-1}\beta_g^{\text{DaRo}}, \tag{4.4}$$

and similarly for h, as the roots there are determined by the equation

$$x^2 - a_p(g)x + \chi(p)p = (x - \alpha_g^{\text{DaRo}})(x - \beta_g^{\text{DaRo}}).$$

In addition, we have $\alpha_g\alpha_h\beta_g\beta_h = p^{-2}\chi_g\chi_h(p)$ and the assumption of [11] is that $\chi_f\chi_g\chi_h = 1$. Finally, our β is their $\beta_f = p/\alpha_{f^*}$. The comparison with their constant $\mathcal{E}_1(f)/\mathcal{E}(f,g,h)$, where the two expressions are given there in equations (6) and (8), should now be easy.

5 A formula for the syntomic regulator on K_2 of curves

In this section, we establish a certain formula for the syntomic regulator for K_2 of curves. Some of the arguments are similar to those in [8], but things need to be redone since the regulator there is written in terms of Coleman integration and not in terms of cup products in syntomic cohomology as required here.

Let $\mathcal{X}$ be a smooth $\mathcal{O}_K$-scheme with generic fiber X and special fiber Y. We recall from [7,8] that there exists a syntomic regulator $\text{reg}_p : K_2(\mathcal{X}) \to H^2_{\text{syn}}(\mathcal{X}, 2)$. We furthermore recall that there exists a (Gros style) modified syntomic cohomology theory $\tilde{H}^*_{\text{ms}}(\mathcal{X}, *)$ with functorial maps $H^*_{\text{syn}}(\mathcal{X}, *) \to \tilde{H}^*_{\text{ms}}(\mathcal{X}, *)$, compatible with cup products. Composing with this we get regulators into modified syntomic cohomology. It further induces an isomorphism $H^2_{\text{syn}}(\mathcal{X}, 2) \cong \tilde{H}^2_{\text{ms}}(\mathcal{X}, 2)$. Thus, for our purposes, we may use these two cohomology theories interchangeably. Modified Gros style syntomic cohomology is related to modified fp-cohomology in the following simple form

$$\tilde{H}^i_{\text{ms}}(\mathcal{X}, n) = \tilde{H}^i_{\text{f},P_n}(\mathcal{X}, n),$$

where P_n is the polynomial $P_n(t) = 1 - t/q^n$ (recall that, as in the setup, q is the cardinality of the residue field and not the coordinate at infinity, as in the modular forms sections). In particular, elements may be represented as in (2.6). The cup products in modified syntomic cohomology are the same as for fp-cohomology, noting that $P_n * P_m = P_{n+m}$. They are compatible with products in K-theory.

Suppose now that $\mathcal{X}$ is proper and of relative dimension 1 over $\mathcal{O}_K$ and let $\eta \in H^1_{\text{dR}}(X/K)$. We will define two regulators associated with η, ultimately showing that they coincide on their common domain.

Definition 5.1 The regulator

$$\text{reg}'_\eta : K_2(K(X))^{t_Y=0} \to K,$$

is defined on the kernel of the tame symbol map along the special fiber Y as follows. The localization sequence in K-theory implies that for each element α in the kernel we may find some open subscheme $\mathcal{Z} \subset \mathcal{X}$, surjecting on $\mathcal{O}_K$, and a (non-unique) element $\beta \in K_2(\mathcal{Z})$ mapping to α. Let $\mathcal{U}$ be the overconvergent space associated with $\mathcal{Z}$. By [8, Corollary 3.6], the element

$$\text{reg}_p(\beta) \in H^2_{\text{syn}}(\mathcal{Z}, 2) \cong H^1_{\text{dR}}(\mathcal{U})$$

depends only on α and we apply the pairing (2.11) with η to obtain $\text{reg}'_\eta(\alpha)$. The result is independent of the choice of $\mathcal{Z}$ by (2.14).

We next define the η regulator

$$\mathrm{reg}_\eta : K_2(K(X)) \to K.$$

We first define it on of a pair of functions as follows:

Definition 5.2 Let $\mathcal{X}$ and η be as above and suppose $f_1, f_2 \in K(X)^\times$. We define

$$\mathrm{reg}_\eta(f_1, f_2) = \langle \eta, (d\log f_1, \gamma_1) \cup (d\log f_2, \gamma_2) \rangle_{\mathcal{U}}.$$

Here the cup product is done in $\tilde{H}^*_{\mathrm{ms}}(\mathcal{Z}, *)$, where $\mathcal{Z}$ is an open subscheme, surjecting on $\mathcal{O}_K$ as before, such that both f_1 and f_2 are invertible on the generic fiber Z and the class $(d\log f_i, \gamma_i) \in \tilde{H}^1_{\mathrm{ms}}(\mathcal{Z}, 1)$ in the representation (2.6) can be any class whose first component is $d\log f_i$ (notice that, implicitly, we are also making a choice of a lift of Frobenius ϕ). The pairing is again with respect to the overconvergent space $\mathcal{U}$ corresponding to $\mathcal{Z}$.

Lemma 5.3 *The η-regulator gives a well-defined map* $\mathrm{reg}_\eta : \bigwedge^2 K(X)^\times \to K$.

Proof The map is clearly bilinear if well defined, and antisymmetric as the cup product in syntomic cohomology is. The choice of $\mathcal{Z}$ does not change anything, by (2.14). Thus, we are free to choose it at our convenience. To show that the regulator is well defined, we need to prove that for any f there exists a $\mathcal{Z}$ such that $d\log f$ can be completed to a class $(d\log f, \gamma) \in \tilde{H}^1_{\mathrm{ms}}(\mathcal{Z}, 1)$. Suppose first that the divisor of f does not contain the special fiber. In this case, for an appropriately chosen $\mathcal{Z}$ we will have $f \in \mathcal{O}(\mathcal{Z})^\times$ and the pair $(d\log f, \log(f_0))$, where $f_0 = f^q/\phi^* f$, is indeed in $\tilde{H}^1_{\mathrm{ms}}(\mathcal{Z}, 1)$ [7]. A general f can be written in the form $\pi^n \tilde{f}$, for some integer n, where $\tilde{f}$ is as above, and since $d\log f = d\log \tilde{f}$ the existence of γ is now clear. To complete the proof, we have to show that the expression for $\mathrm{reg}_\eta(f_1, f_2)$ is independent of the choices of γ_1 and γ_2. This is because, by (2.5), for a constant c, we have $(0, c) \cup d\log f, \gamma) = c d\log f$ and $\langle \eta, d\log f \rangle = 0$ by [8, Lemma 4.9]. Note, finally, that since the choices of γ_1 and γ_2 do not matter, the expression is also independent of the choice of the lift of Frobenius ϕ. □

As a side effect of the above proof, we obtain the fact that

$$\mathrm{reg}_\eta(c, f) = 0 \text{ for a constant function } c. \tag{5.1}$$

Proposition 5.4 *The maps* reg_η *induces a well-defined map* $\mathrm{reg}_\eta : K_2(K(X)) \to K$.

Proof We need to prove that $\mathrm{reg}_\eta(f, 1-f) = 0$ for any rational function f. This is true if the divisors of both f and $1-f$ do not contain the special fiber. Indeed, taking an open $\mathcal{Z}$ on which both f and $1-f$ are invertible, the expression $(d\log f_1, \gamma_1) \cup (d\log f_2, \gamma_2)$, for appropriate γ_i, is nothing but the syntomic regulator of the element $(f) \cup (1-f) \in K_2(\mathcal{Z})$. This element maps to 0 in $K_2(K(X))$; hence, the above regulator is 0 by [8, Corollary 3.6]. The η-regulator is then 0 as well.

It remains to check the case $f = \pi^{\pm n} g$, with n positive and where the divisor of g does not contain the special fiber (when the divisor of f contains Y we switch the roles of f and $1-f$). Consider first $\mathrm{reg}_\eta(\pi^n g, 1 - \pi^n g)$. We will need the following.

Lemma 5.5 *Suppose $\mathcal{X}$ is $\mathbb{P}^1$ localized at $0, \pi^{-n}$ and ∞ with standard coordinate t and consider the element $\alpha = (t) \cup (1-\pi^n t) \in K_2(\mathcal{X})$. Then, $\mathrm{reg}_p(\alpha) = 0 \in \tilde{H}^2_{\mathrm{ms}}(\mathcal{X}, 2)$.*

Proof We have $\tilde{H}^2_{\mathrm{ms}}(\mathcal{X}, 2) = H^1_{\mathrm{rig}}(\mathbb{P}^1 - \{0, \infty\})$. As $H_{\mathrm{rig}}(\mathbb{P}^1) = 0$ it suffices to show that the residues of $\mathrm{reg}_p(\alpha)$ at 0 and ∞ are 0. These residues are in turn the logs of the products of the tame symbols inside the corresponding residue disks [1]. The tame symbols are 1 at $0, \pi^n$ at ∞ and π^{-n} at π^{-n}. The result is thus clear. □

Going back to the proof of the Proposition, it follows immediately from the lemma by pullback that $\mathrm{reg}_\eta(g, 1-\pi^n g) = 0$. Bilinearity and (5.1) finish the proof of this first case. Similarly, we have, using (5.1),

$$\begin{aligned}\mathrm{reg}_\eta(\pi^{-n}g, 1-\pi^{-n}g) &= \mathrm{reg}_\eta(g, 1-\pi^{-n}g) = \mathrm{reg}_\eta(g, \pi^n - g)\\ &= \mathrm{reg}_\eta(g, g) + \mathrm{reg}_\eta(g, \pi^n g^{-1} - 1) = \mathrm{reg}_\eta(g, \pi^n g^{-1} - 1)\end{aligned}$$

where we have used the antisymmetry of the regulator

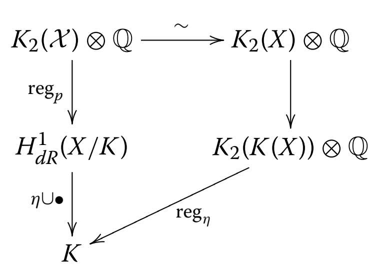
$$\begin{aligned}&= \mathrm{reg}_\eta(g, 1-\pi^n g^{-1})\\ &= -\mathrm{reg}_\eta(g^{-1}, 1-\pi^n g^{-1})\\ &= -\mathrm{reg}_\eta(\pi^n g^{-1}, 1-\pi^n g^{-1}) = 0\end{aligned}$$

by the previous case. □

Proposition 5.6 *The two η-regulators coincide on $K_2(K(X))^{t_Y=0}$.*

Proof This follows because by the proof of Theorem 3 in [8] an element of $K_2(K(X))^{t_Y=0}$ may be written as a sum of symbols $\{f, g\}$, where $f, g \in \mathcal{O}(\mathcal{Z})^\times$. For some $\mathcal{Z}$ and for these, the equality of the two regulators follows essentially by definition. □

Corollary 5.7 *The diagram*



commutes

Proof Indeed, elements of $K_2(\mathcal{X})$ map to the kernel of t_Y in $K_2(K(X))$ and, essentially by definition, the composed down arrow is just reg'_η. □

6 Other formulas

In this section, we establish the formulas in [2] and [4]. The proofs are now given by roughly the same computations as before, slightly simplified.

We begin with [2]. They consider modular units u_1 and u_2 related to certain weight 2 Eisenstein series,

$$d\log u_1 = E_{2,\chi}(q)\frac{\mathrm{d}q}{q} := \omega_1, \quad d\log u_2 = E_2(\chi_1,\chi_2)(q)\frac{\mathrm{d}q}{q} := \omega_2, \tag{6.1}$$

with $\chi^{-1} = \chi_1\chi_2$. Note that the forms associated with the Eisenstein series have logarithmic poles at the cusps. The Steinberg symbol $\{u_1, u_2\}$, apriorilly in K_2 of the function field of $X_1(N)$, in fact extends to $K_2(X_1(N))$. We will let K be a sufficiently large extension of $\mathbb{Q}_p$ so that the symbols are defined over K. The following theorem appears as formula (60) in [2].

Theorem 6.1 *Let η, β and α be as in Theorem 3.3. For any lift $\tilde{u} \in K_2(X_1(N))$ of the symbol $\{u_1, u_2\}$ the image of $\tilde{u}$ under*

$$K_2(X_1(N))\otimes\mathbb{Q} \cong K_2(\mathcal{X}_1(N))\otimes\mathbb{Q} \xrightarrow{\mathrm{reg}_p} H^1_{dR}(X_1(N)/K) \xrightarrow{\eta\cup\bullet} K$$

equals

$$\begin{aligned}&(1-\beta^2p^{-2})\\ &\quad\times(1-\beta p^{-1}\chi_1(p)\chi(p))(1-\beta\chi_2(p)\chi(p))(1-\beta p^{-2}\chi_1(p))(1-\beta\chi_2(p)p^{-1})\\ &\quad\times\left\langle \eta, e_{\mathrm{ord}}(\theta^{-1}E^{[p]}_{2,\chi}\omega_2)\right\rangle.\end{aligned}$$

Proof By Corollary 5.7 this boils down to the computation of $\mathrm{reg}_\eta(\{u_1,u_2\})$. By Definition 5.2 we need to lift ω_1 and ω_2 to classes in fp-cohomology, in any way we choose, compute their cup product and cup with η. Note that $E_{2,\chi}$ is an eigenvector for T_p with eigenvalue $1+\chi(p)p$ so that we get the equation

$$(1-\chi(p)\phi^*)(1-p^{-1}\phi^*)\omega_1 = \mathrm{d}\theta^{-1}E^{[p]}_{2,\chi},$$

so the roots of the characteristic equation are $\chi(p)$ and p^{-1}. Similarly, for $E_2(\chi_1,\chi_2)$ the roots are $\chi_2(p))$ and $\chi_1(p)p^{-1}$. The result therefore follows from the Eisenstein series case of Corollary 3.4. □

Remark 6.2 For the comparison with the results of [2] we note that they assume that $\chi_f = 1$ and that $\chi\chi_1\chi_2 = 1$. The relevant constants appear in Proposition 3.2 and the formula immediately following it with $k=\ell=2$. However, we get the formula with their $\mathcal{E}(f)$ rather than $\mathcal{E}^*(f)$!!

Next, consider [4]. This paper concerns the syntomic regulator

$$\mathrm{reg}_{\mathrm{syn}} : CH^2(S,1) \to H^2_{\mathrm{dR}}(S/K)/F^2 \cong \mathrm{Hom}(F^1H^2_{\mathrm{dR}}(S/K), K),$$

where S is the self-product of $X_1(N)$ and the field of coefficients extended to some p-adic field K. Recall that an element of $CH^2(S,1)$ for the surface S consists of a formal sum $\sum(C_i, f_i)$ where the C_i are curves on S and f_i is a rational functions on C_i such that the sum of the divisors of the f_i vanishes on S. This regulator was treated in detail in [9]. The paper [4] does not use the final formulas of [9] but one can derive the required results from the proofs there.

The case considered in [4] is the following: the surface is $S = X_1(N) \times X_1(N)$, with projections π_i, $i = 1, 2$, on $X_1(N)$, and the element is $\Theta = (\Delta, u) + \sum(C_i, f_i)$. Here, Δ is the diagonal and u is defined in [4, Definition 2.4] to be the modular unit with $d\log(u) = E_{2,\chi}$, normalized so that its value at the cusp ∞ is 1 in the sense of Brunault [10, Section 5]. This means that the coefficient of lowest power of q in its q-expansion is 1 (note that unlike the K_2 case, the precise normalization of the unit is important). Finally, the C_i are either horizontal or vertical curves with functions f_i arranged in such a way as to make Θ an element of $CH^2(S, 1)$. We will not need to know these terms, called *negligible* in [4], precisely, as their contribution to the regulator will vanish. We will prove the following result of [4], which is equation (4.2) there.

Theorem 6.3 *Let $\eta, \alpha, \beta, g, \alpha_g$ and β_g be as in Corollary 3.4. Then $\operatorname{reg}_{syn}(\Theta)$, viewed as an element of $\operatorname{Hom}(F^1H^2_{dR}(S/K), K)$, evaluates on $\pi_1^*(\omega_g) \cup \pi_2^*(\eta)$ to give*

$$
\begin{aligned}
&(1 - \beta^2\alpha_g\beta_g\chi(p)p^{-1}) \\
&\quad\times \left((1 - \beta\alpha_g\chi(p))(1 - \beta\beta_g\chi(p))(1 - \beta\alpha_g p^{-1})(1 - \beta\beta_g p^{-1})\right)^{-1} \\
&\quad\times \left\langle \eta, e_{\mathrm{ord}}(\theta^{-1}E^{[p]}_{2,\chi}\omega_g)\right\rangle.
\end{aligned}
$$

Proof According to [9, Theorem 1.1], in order to compute the regulator one first picks up Coleman integrals F_{ω_g} and F_η, which yield via pullbacks, integrals $\pi_1^*F_{\omega_g}$ and $\pi_2^*F_\eta$ to $\pi_1^*\omega_g$ and $\pi_2^*\eta$, respectively. As the integrality assumption 1 of the theorem is satisfied, as noted in [4, p. 371], the regulator is a sum of terms corresponding to the summands in Θ. We consider the term corresponding to (Δ, u) separately. The other terms are computed using "global triple indices": $\langle F_\eta|_{C_i}, \log(f_i); F_\omega|_{C_i}\rangle_{\mathrm{gl},X_i}$. We do not need to get into the definitions here other than to point out that because the curves C_i are either vertical or horizontal in all the terms either the first or last coordinate will be constant, and they therefore vanish using [3, Lemma 8.3 and Proposition 8.4]. □

It remains to compute the term corresponding to (Δ, u). The formula we are after is hidden in the proof of [9, Theorem 1.1]. We make it explicit as follows:

Lemma 6.4 *Let S/K be a surface and let $\Theta = \sum(Z_i, f_i) \in CH^2(S, 1)$. Let $g_i : X_i \to S$ be the normalizations of the Z_i. Let $\omega \in F^1H^1_{dR}(S/K)$ and $[\eta] \in H^1_{dR}(S/K)$ represented by the form of the second kind η. Pick a Coleman integral F_ω to ω. Then, under the integrality assumption, $\operatorname{reg}_{syn}(\Theta)$ evaluated on $\omega \cup [\eta]$ is a sum of terms. The term corresponding to (Z_i, f_i) can be computed as follows: Let $\tilde{\omega}$ be the class in $H^1_{\mathrm{fp},\{2\}}(\mathcal{X}, 1)$ corresponding to $g_i^*F_\omega$ and abuse notation to let η be $g_i^*\eta$. Then, the relevant term is*

$$\langle \eta, \operatorname{reg}(f) \cup \tilde{\omega}\rangle,$$

where $\operatorname{reg}(f) \cup \tilde{\omega}$ is computed in fp-cohomology of some open subscheme of $\mathcal{X}_i$, then identified with the first de Rham cohomology of some wide open in X_i.

Proof This expression is derived along the proof of Theorem 1.1 in [9]. The relevant computation is done on page 62, with the key formula being the last displayed formula on that page. The expression obtained there is then reformulated in equation (6.5) there, and the proof is complete by noting that $\langle F_\eta, F_\omega\rangle_{gl} = \langle \eta, \omega\rangle$, for the pairing defined in (2.11). This last fact is proved in [8, Proposition 4.10]. □

Going back to the proof of the theorem, since $\pi_i \circ \Delta = Id$, the above Lemma immediately gives the following expression for the remaining term:

$$\langle \eta, \mathrm{reg}(u) \cup \tilde{\omega}_g \rangle.$$

Using Corollary 3.4 finishes the proof. One only needs to note, since we are dealing with an Eisenstein series, that the Brunault normalization implies that $\log(u)$ has constant term 0 at ∞ with respect to the parameter q.

Remark 6.5 For the comparison with [4], the relevant constants are given in Proposition 2.7 there, with $k = m = 2$ and $t = -1$. For the comparison note that they assume $\chi = \chi_f^{-1}\chi_g^{-1}$ (following (2.2) there) and that $\alpha_g\beta_g = p^{-1}\chi_g(p)$. Also recall (4.4) and the relation $\beta = \beta_f$ as before.

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