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Bogomolov's proof of the geometric version of the Szpiro Conjecture from the point of view of inter-universal Teichmüller theory

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Abstract

The purpose of the present paper is to expose, in substantial detail, certain **remarkable similarities** between **inter-universal Teichmüller theory** and the theory surrounding **Bogomolov's proof** of the **geometric** version of the **Szpiro Conjecture**. These similarities are, in some sense, consequences of the fact that both theories are closely related to the hyperbolic geometry of the classical **upper half-plane**. We also discuss various differences between the theories, which are closely related to the *conspicuous absence* in Bogomolov's proof of **Gaussian distributions** and **theta functions**, i.e., which play a central role in inter-universal Teichmüller theory.

Keywords: Bogomolov's proof, Szpiro Conjecture, Hyperbolic geometry, Symplectic geometry, Upper half-plane, Theta function, Gaussian distribution, Inter-universal Teichmüller theory, Multiradial representation, Indeterminacies

Contents

1 Background

Certain aspects of the inter-universal Teichmüller theory developed in [6–9]—namely

(IU1) the **geometry** of $\Theta^{\pm \mathrm{ell}}$**NF-Hodge theaters** (cf. [6, Definition 6.13]; [6, Remark 6.12.3]),

(IU2) the **precise** relationship between **arithmetic degrees**—i.e., of *q-pilot* and Θ*-pilot* objects—given by the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$**-link** (cf. [8, Definition 3.8, (i), (ii)]; [8, Remark 3.10.1, (ii)]), and

(IU3) the **estimates** of log-volumes of certain subsets of **log-shells** that give rise to **diophantine inequalities** (cf. [9, §1, §2]; [8, Remark 3.10.1, (iii)]) such as the **Szpiro Conjecture**

—are *substantially reminiscent* of the theory surrounding **Bogomolov's proof** of the **geometric** version of the **Szpiro Conjecture**, as discussed in [1,10]. Put another way, these aspects of inter-universal Teichmüller theory may be thought of as **arithmetic analogues** of the geometric theory surrounding Bogomolov's proof. Alternatively, Bogomolov's proof may be thought of as a sort of **useful elementary guide**, or **blueprint** [perhaps even a sort of **Rosetta stone**!], for understanding substantial portions of inter-universal Teichmüller theory. The author would like to express his gratitude to *Ivan Fesenko* for bringing to his attention, via numerous discussions in person, e-mails, and skype conversations between December 2014 and January 2015, the possibility of the existence of such fascinating connections between Bogomolov's proof and inter-universal Teichmüller theory.

After reviewing, in Sects. 2–4, the theory surrounding Bogomolov's proof from a point of view that is somewhat closer to inter-universal Teichmüller theory than the point of view of [1,10], we then proceed, in Sects. 5 and 6, to *compare*, by highlighting various *similarities* and *differences*, Bogomolov's proof with inter-universal Teichmüller theory. In a word, the similarities between the two theories revolve around the relationship of both theories to the *classical elementary geometry of the* ***upper half-plane***, while the differences between the two theories are closely related to the ***conspicuous absence*** *in Bogomolov's proof of* ***Gaussian distributions*** *and* ***theta functions***, i.e., which play a central role in inter-universal Teichmüller theory.

2 The geometry surrounding Bogomolov's proof

First, we begin by reviewing the *geometry* surrounding Bogomolov's proof, albeit from a point of view that is somewhat more abstract and conceptual than that of [1,10].

We denote by $\mathcal{M}$ the *complex analytic moduli stack of elliptic curves* (i.e., one-dimensional complex tori). Let

$$\widetilde{\mathcal{M}} \rightarrow \mathcal{M}$$

be a *universal covering* of $\mathcal{M}$. Thus, $\widetilde{\mathcal{M}}$ is non-canonically isomorphic to the **upper half-plane** $\mathfrak{H}$. In the following, we shall denote by a *subscript* $\widetilde{\mathcal{M}}$ the result of restricting to $\widetilde{\mathcal{M}}$ objects over $\mathcal{M}$ that are denoted by a subscript $\mathcal{M}$.

Write

$$\omega_{\mathcal{M}} \rightarrow \mathcal{M}$$

for the [geometric!] line bundle determined by the *cotangent space at the origin* of the tautological family of elliptic curves over $\mathcal{M}$; $\omega^{\times}_{\mathcal{M}} \subseteq \omega_{\mathcal{M}}$ for the complement of the zero section in $\omega_{\mathcal{M}}$; $\mathcal{E}_{\mathcal{M}}$ for the local system over $\mathcal{M}$ determined by the *first singular cohomology modules with coefficients in* $\mathbb{R}$ of the fibers over $\mathcal{M}$ of the tautological family of elliptic curves over $\mathcal{M}$; $\mathcal{E}^{\times}_{\mathcal{M}} \subseteq \mathcal{E}_{\mathcal{M}}$ for the complement of the zero section in $\mathcal{E}_{\mathcal{M}}$. Thus, if we think of bundles as geometric spaces/stacks, then there is a *natural embedding*

$$\omega_{\mathcal{M}} \hookrightarrow \mathcal{E}_{\mathcal{M}} \otimes_{\mathbb{R}} \mathbb{C}$$

(cf. the inclusion "$\omega \hookrightarrow \mathcal{E}$" of [6, Remark 4.3.3, (ii)]). Moreover, this natural embedding, together with the **natural symplectic form**

$$\langle \text{-} , \text{-} \rangle_{\mathcal{E}}$$

on $\mathcal{E}_{\mathcal{M}}$ [i.e., determined by the *cup product* on the singular cohomology of fibers over $\mathcal{M}$, together with the *orientation* that arises from the complex holomorphic structure on these fibers], gives rise to a **natural metric** (cf. the discussion of [6, Remark 4.3.3, (ii)]) on $\omega_{\mathcal{M}}$. Write

$$(\omega_{\mathcal{M}} \supseteq \omega_{\mathcal{M}}^{\times} \supseteq) \quad \omega_{\mathcal{M}}^{\measuredangle} \to \mathcal{M}$$

for the $\mathbb{S}^1$-bundle over $\mathcal{M}$ determined by the points of $\omega_{\mathcal{M}}$ of **modulus one** with respect to this natural metric.

Next, observe that the natural section $\frac{1}{2} \cdot \mathrm{tr}(-) : \mathbb{C} \to \mathbb{R}$ (i.e., one-half the trace map of the Galois extension $\mathbb{C}/\mathbb{R}$) of the natural inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ determines a *section* $\mathcal{E}_{\mathcal{M}} \otimes_{\mathbb{R}} \mathbb{C} \to \mathcal{E}_{\mathcal{M}}$ of the natural inclusion $\mathcal{E}_{\mathcal{M}} \hookrightarrow \mathcal{E}_{\mathcal{M}} \otimes_{\mathbb{R}} \mathbb{C}$ whose *restriction* to $\omega_{\mathcal{M}}$ determines *bijections*

$$\omega_{\mathcal{M}} \overset{\sim}{\to} \mathcal{E}_{\mathcal{M}}, \quad \omega_{\mathcal{M}}^{\times} \overset{\sim}{\to} \mathcal{E}_{\mathcal{M}}^{\times}$$

(i.e., of geometric bundles over $\mathcal{M}$). Thus, at the level of *fibers*, the bijection $\omega_{\mathcal{M}} \overset{\sim}{\to} \mathcal{E}_{\mathcal{M}}$ may be thought of as a (non-canonical) copy of the natural bijection $\mathbb{C} \overset{\sim}{\to} \mathbb{R}^2$.

Next, let us write E for the *fiber* (which is non-canonically isomorphic to $\mathbb{R}^2$) of the local system $\mathcal{E}_{\mathcal{M}}$ relative to some *basepoint* corresponding to a **cusp**

"∞"

of $\widetilde{\mathcal{M}}$, $E_{\mathbb{C}} \overset{\mathrm{def}}{=} E \otimes_{\mathbb{R}} \mathbb{C}$, $SL(E)$ for the group of $\mathbb{R}$-linear automorphisms of E that preserve the *natural symplectic form* $\langle -, - \rangle_E \overset{\mathrm{def}}{=} \langle -, - \rangle_{\mathcal{E}}|_E$ on E [so $SL(E)$ is non-canonically isomorphic to $SL_2(\mathbb{R})$]. Now since $\widetilde{\mathcal{M}}$ is *contractible*, the local systems $\mathcal{E}_{\widetilde{\mathcal{M}}}$, $\mathcal{E}_{\widetilde{\mathcal{M}}}^{\times}$ over $\widetilde{\mathcal{M}}$ are *trivial*. In particular, we obtain **natural projection maps**

$$\mathcal{E}_{\widetilde{\mathcal{M}}} \twoheadrightarrow E, \quad \mathcal{E}_{\widetilde{\mathcal{M}}}^{\times} \twoheadrightarrow E^{\times} \twoheadrightarrow E^{\angle} \twoheadrightarrow E^{|\angle|}$$

—where we write

$$E^{\times} \overset{\mathrm{def}}{=} E \backslash \{(0,0\}, \quad E^{\angle} \overset{\mathrm{def}}{=} E^{\times}/\mathbb{R}_{>0}$$

[so $E^{\times}, E^{\angle}$ are non-canonically isomorphic to $\mathbb{R}^{2\times} \overset{\mathrm{def}}{=} \mathbb{R}^2 \backslash \{(0,0)\}$, $\mathbb{R}^{2\angle} \overset{\mathrm{def}}{=} \mathbb{R}^{2\times}/\mathbb{R}_{>0} \cong \mathbb{S}^1$, respectively] and

$$E^{\angle} \twoheadrightarrow E^{|\angle|} \overset{\mathrm{def}}{=} E^{\angle}/\{\pm 1\}$$

for the finite étale covering of degree 2 determined by forming the quotient by the action of $\pm 1 \in SL(E)$.

Next, let us observe that over each point $\widetilde{\mathcal{M}}$, the composite

$$\omega_{\widetilde{\mathcal{M}}}^{\measuredangle} \subseteq \omega_{\widetilde{\mathcal{M}}}^{\times} \overset{\sim}{\to} \mathcal{E}_{\widetilde{\mathcal{M}}}^{\times} \twoheadrightarrow E^{\times} \twoheadrightarrow E^{\angle}$$

induces a *homeomorphism* between the fiber of $\omega_{\widetilde{\mathcal{M}}}^{\measuredangle}$ [over the given point of $\widetilde{\mathcal{M}}$] and $E^{\angle}$. In particular, for each point of $\widetilde{\mathcal{M}}$, the metric on this fiber of $\omega_{\widetilde{\mathcal{M}}}^{\measuredangle}$ determines a **metric** on $E^{\angle}$ (i.e., which *depends* on the point of $\widetilde{\mathcal{M}}$ under consideration!). On the other hand, one verifies immediately that such metrics on $E^{\angle}$ always satisfy the following property: Let

$$\overline{D}^{\angle} \subseteq E^{\angle}$$

be a **fundamental domain** for the action of ± 1 on $E^{\angle}$, i.e., the closure of some open subset $D^{\angle} \subseteq E^{\angle}$ such that $D^{\angle}$ maps *injectively* to $E^{|\angle|}$, while $\overline{D}^{\angle}$ maps *surjectively* to $E^{|\angle|}$. Thus, $\pm\overline{D}^{\angle}$ (i.e., the $\{\pm 1\}$-orbit of $\overline{D}^{\angle}$) is equal to $E^{\angle}$. Then the **volume** of $\overline{D}^{\angle}$ relative to metrics on $E^{\angle}$ of the sort just discussed is always equal to π, while the **volume** of $\pm\overline{D}^{\angle}$ (i.e., $E^{\angle}$) relative to such a metric is always equal to 2π.

Over each point of $\widetilde{\mathcal{M}}$, the composite $\omega^{\times}_{\widetilde{\mathcal{M}}} \xrightarrow{\sim} \mathcal{E}^{\times}_{\widetilde{\mathcal{M}}} \twoheadrightarrow E^{\times}$ corresponds (non-canonically) to a copy of the natural bijection $\mathbb{C}^{\times} \xrightarrow{\sim} \mathbb{R}^{2\times}$ that arises from the **complex structure** on E determined by the point of $\widetilde{\mathcal{M}}$. Moreover, this assignment of complex structures, or, alternatively, points of the one-dimensional complex projective space $\mathbb{P}(E_{\mathbb{C}})$, to points of $\widetilde{\mathcal{M}}$ determines a *natural embedding*

$$\widetilde{\mathcal{M}} \hookrightarrow \mathbb{P}(E_{\mathbb{C}})$$

—i.e., a copy of the usual embedding of the **upper half-plane** into the complex projective line—hence also *natural actions* of $SL(E)$ on $\widetilde{\mathcal{M}}$ and $\mathcal{E}_{\widetilde{\mathcal{M}}}$ that are *uniquely determined* by the property that they be *compatible*, relative to this natural embedding and the projection $\mathcal{E}_{\widetilde{\mathcal{M}}} \twoheadrightarrow E$, with the natural actions of $SL(E)$ on $\mathbb{P}(E_{\mathbb{C}})$ and E. One verifies immediately that these natural actions also determine *compatible* natural actions of $SL(E)$ on $\omega^{\measuredangle}_{\widetilde{\mathcal{M}}} \subseteq \omega^{\times}_{\widetilde{\mathcal{M}}} \xrightarrow{\sim} \mathcal{E}^{\times}_{\widetilde{\mathcal{M}}}$ and that the natural action of $SL(E)$ on $\omega^{\measuredangle}_{\widetilde{\mathcal{M}}}$ determines a structure of $SL(E)$-**torsor** on $\omega^{\measuredangle}_{\widetilde{\mathcal{M}}}$. Also, we observe that the natural embedding of the above display allows one to regard $E^{|\angle|}$ as the **"boundary"** $\partial\widetilde{\mathcal{M}}$ of $\widetilde{\mathcal{M}}$, i.e., the boundary of the upper half-plane.

Let $\widetilde{SL}(E)$, $(\omega^{\measuredangle}_{\mathcal{M}})^{\sim}$, $(\omega^{\times}_{\mathcal{M}})^{\sim}$, $(\mathcal{E}^{\times}_{\mathcal{M}})^{\sim}$, $(E^{\times})^{\sim}$, $(E^{\angle})^{\sim}$ be *compatible universal coverings* of $SL(E)$, $\omega^{\measuredangle}_{\widetilde{\mathcal{M}}}$, $\omega^{\times}_{\widetilde{\mathcal{M}}}$, $\mathcal{E}^{\times}_{\widetilde{\mathcal{M}}}$, $E^{\times}$, $E^{\angle}$, respectively. Thus, $\widetilde{SL}(E)$ admits a natural Lie group structure, together with a *natural surjection of Lie groups* $\widetilde{SL}(E) \twoheadrightarrow SL(E)$, whose kernel admits a *natural generator*

$$\widetilde{\tau}^{\measuredangle} \in \operatorname{Ker}\left(\widetilde{SL}(E) \twoheadrightarrow SL(E)\right)$$

determined by the *clockwise orientation* that arises from the *complex structure* on the fibers of $\omega^{\times}_{\mathcal{M}}$ over $\mathcal{M}$]. This natural generator determines a natural isomorphism $\mathbb{Z} \xrightarrow{\sim} \operatorname{Ker}(\widetilde{SL}(E) \twoheadrightarrow SL(E))$.

Next, observe that the natural actions of $SL(E)$ on $\omega^{\measuredangle}_{\widetilde{\mathcal{M}}}$, $\omega^{\times}_{\widetilde{\mathcal{M}}}$, $\mathcal{E}^{\times}_{\widetilde{\mathcal{M}}}$, $E^{\times}$, $E^{\angle}$ lift uniquely to compatible natural actions of $\widetilde{SL}(E)$ on the respective universal coverings $(\omega^{\measuredangle}_{\mathcal{M}})^{\sim}$, $(\omega^{\times}_{\mathcal{M}})^{\sim}$, $(\mathcal{E}^{\times}_{\mathcal{M}})^{\sim}$, $(E^{\times})^{\sim}$, $(E^{\angle})^{\sim}$. In particular, the natural generator $\widetilde{\tau}^{\measuredangle}$ of $\mathbb{Z} = \operatorname{Ker}(\widetilde{SL}(E) \twoheadrightarrow SL(E))$ determines a natural generator $\widetilde{\tau}^{\angle}$ of the group $\operatorname{Aut}((E^{\angle})^{\sim}/E^{\angle})$ of covering transformations of $(E^{\angle})^{\sim} \twoheadrightarrow E^{\angle}$ and hence, taking into account the composite covering $(E^{\angle})^{\sim} \twoheadrightarrow E^{\angle} \twoheadrightarrow E^{|\angle|}$, a **natural** $\operatorname{Aut}_{\pi}(\mathbb{R})$**-orbit** of **homeomorphisms** [i.e., a "homeomorphism that is well defined up to composition with an element of $\operatorname{Aut}_{\pi}(\mathbb{R})$"]

$$\left(E^{\angle}\right)^{\sim} \xrightarrow{\sim} \mathbb{R} \quad (\curvearrowleft \operatorname{Aut}_{\pi}(\mathbb{R}))$$

—where we write $\operatorname{Aut}_{\pi}(\mathbb{R})$ for the group of self-homeomorphisms $\mathbb{R} \xrightarrow{\sim} \mathbb{R}$ that *commute* with translation by π. Here, the group of covering transformations of the covering $(E^{\angle})^{\sim} \twoheadrightarrow E^{\angle}$ is generated by the transformation $\widetilde{\tau}^{\angle}$, which corresponds to **translation** by 2π; the group of covering transformations of the composite covering $(E^{\angle})^{\sim} \twoheadrightarrow E^{\angle} \twoheadrightarrow E^{|\angle|}$ admits a generator $\widetilde{\tau}^{|\angle|}$ that satisfies the relation

$$\left(\widetilde{\tau}^{|\angle|}\right)^2 = \widetilde{\tau}^{\angle} \in \operatorname{Aut}\left(\left(E^{\angle}\right)^{\sim} \Big/ E^{\angle}\right)$$

and corresponds to **translation** by π (cf. the transformation "$z(-)$" of [10, Lemma 3.5]). Moreover, $\widetilde{\tau}^{|\angle|}$ arises from an element $\widetilde{\tau}^{|\measuredangle|} \in \widetilde{SL}(E)$ that *lifts* $-1 \in SL(E)$ and satisfies the relation $(\widetilde{\tau}^{|\measuredangle|})^2 = \widetilde{\tau}^{\measuredangle}$. The geometry discussed so far is summarized in the commutative diagram of Fig. 1.

$$
\begin{array}{ccccccccccccc}
\widetilde{SL}(E) \curvearrowright & & & & & & & & & & & & (\xrightarrow{\sim} \mathbb{R}) \\
\Big\{ & & (\omega^{\angle}_{\mathcal{M}})^{\sim} & \subseteq & (\omega^{\times}_{\mathcal{M}})^{\sim} & \xrightarrow{\sim} & (\mathcal{E}^{\times}_{\mathcal{M}})^{\sim} & \twoheadrightarrow & (E^{\times})^{\sim} & \twoheadrightarrow & (E^{\angle})^{\sim}\Big\} \\
SL(E) \curvearrowright & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Big\{\widetilde{\mathcal{M}} & \longleftarrow & \omega^{\angle}_{\widetilde{\mathcal{M}}} & \subseteq & \omega^{\times}_{\widetilde{\mathcal{M}}} & \xrightarrow{\sim} & \mathcal{E}^{\times}_{\widetilde{\mathcal{M}}} & \twoheadrightarrow & E^{\times} & \twoheadrightarrow & E^{\angle}\Big\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & (\cong \mathbb{R}^{2\times}) & & (\cong \mathbb{R}^{2\angle} \\
\mathcal{M} & \longleftarrow & \omega^{\angle}_{\mathcal{M}} & \subseteq & \omega^{\times}_{\mathcal{M}} & \xrightarrow{\sim} & \mathcal{E}^{\times}_{\mathcal{M}} & & & & \cong \mathbb{S}^1)
\end{array}
$$

Fig. 1 The geometry surrounding Bogomolov's proof

3 Fundamental groups in Bogomolov's proof

Next, we discuss the various **fundamental groups** that appear in Bogomolov's proof.

Recall that the 12th tensor power $\omega_{\mathcal{M}}^{\otimes 12}$ of the line bundle $\omega_{\mathcal{M}}$ admits a natural section; namely, the so-called **discriminant modular form**, which is *nonzero* over $\mathcal{M}$, hence determines a section of $\omega_{\mathcal{M}}^{\times\otimes 12}$ (i.e., the complement of the zero section of $\omega_{\mathcal{M}}^{\otimes 12}$). Thus, by raising sections of $\omega^{\times}_{\mathcal{M}}$ to the 12th power and then applying the *trivialization* determined by the discriminant modular form, we obtain **natural holomorphic surjections**

$$\omega^{\times}_{\mathcal{M}} \twoheadrightarrow \omega_{\mathcal{M}}^{\times\otimes 12} \twoheadrightarrow \mathbb{C}^{\times}$$

—where we note that the first surjection $\omega^{\times}_{\mathcal{M}} \twoheadrightarrow \omega_{\mathcal{M}}^{\times\otimes 12}$, as well as the pull-back $\omega^{\times}_{\widetilde{\mathcal{M}}} \twoheadrightarrow \omega_{\widetilde{\mathcal{M}}}^{\times\otimes 12}$ of this surjection to $\widetilde{\mathcal{M}}$, is in fact a *finite étale covering* of complex analytic stacks. Thus, the universal covering $(\omega^{\times}_{\mathcal{M}})^{\sim}$ over $\omega^{\times}_{\mathcal{M}}$ may be regarded as a universal covering $(\omega_{\mathcal{M}}^{\times\otimes 12})^{\sim} \stackrel{\mathrm{def}}{=} (\omega^{\times}_{\mathcal{M}})^{\sim}$ of $\omega_{\mathcal{M}}^{\times\otimes 12}$. In particular, if we regard $\mathbb{C}$ as a universal covering of $\mathbb{C}^{\times}$ via the *exponential map* $\exp : \mathbb{C} \twoheadrightarrow \mathbb{C}^{\times}$, then the surjection $\omega_{\mathcal{M}}^{\times\otimes 12} \twoheadrightarrow \mathbb{C}^{\times}$ determined by the discriminant modular form lifts to a surjection

$$\left(\omega^{\times}_{\mathcal{M}}\right)^{\sim} = \left(\omega_{\mathcal{M}}^{\times\otimes 12}\right)^{\sim} \twoheadrightarrow \mathbb{C}$$

of universal coverings that is well defined up to composition with a covering transformation of the universal covering $\exp : \mathbb{C} \twoheadrightarrow \mathbb{C}^{\times}$.

Next, let us recall that the $\mathbb{R}$-vector space E is equipped with a *natural* $\mathbb{Z}$*-lattice*

$$E_{\mathbb{Z}} \subseteq E$$

(i.e., determined by the *singular cohomology with coefficients in* $\mathbb{Z}$). The set of elements of $SL(E)$ that *stabilize* $E_{\mathbb{Z}} \subseteq E$ determines a subgroup $SL(E_{\mathbb{Z}}) \subseteq SL(E)$ [so $SL(E_{\mathbb{Z}})$ is non-canonically isomorphic to $SL_2(\mathbb{Z})$], hence also a subgroup $\widetilde{SL}(E_{\mathbb{Z}}) \stackrel{\mathrm{def}}{=} \widetilde{SL}(E) \times_{SL(E)} SL(E_{\mathbb{Z}})$. Thus, $SL(E) \supseteq SL(E_{\mathbb{Z}})$ admits a *natural action* on $\omega^{\times}_{\widetilde{\mathcal{M}}}$; $\widetilde{SL}(E) \supseteq \widetilde{SL}(E_{\mathbb{Z}})$ admits a *natural action* on $(\omega^{\times}_{\mathcal{M}})^{\sim}$. Moreover, one verifies immediately that the latter natural action determines a **natural isomorphism**

$$\widetilde{SL}(E_{\mathbb{Z}}) \xrightarrow{\sim} \pi_1\left(\omega^{\times}_{\mathcal{M}}\right)$$

with the *group of covering transformations* of $(\omega^{\times}_{\mathcal{M}})^{\sim}$ over $\omega^{\times}_{\mathcal{M}}$, i.e., with the *fundamental group* [relative to the basepoint corresponding to the universal covering $(\omega^{\times}_{\mathcal{M}})^{\sim}$] $\pi_1(\omega^{\times}_{\mathcal{M}})$.

In particular, if we use the *generator* $-2\pi i \in \mathbb{C}$ to identify $\pi_1(\mathbb{C}^{\times})$ with $\mathbb{Z}$, then one verifies easily (by considering the complex elliptic curves that admit automorphisms of order >2) that we obtain a **natural surjective homomorphism**

$$\chi : \widetilde{SL}(E_{\mathbb{Z}}) = \pi_1\left(\omega^{\times}_{\mathcal{M}}\right) \twoheadrightarrow \pi_1\left(\mathbb{C}^{\times}\right) \xrightarrow{\sim} \mathbb{Z}$$

whose restriction to $\mathbb{Z} \xrightarrow{\sim} \mathrm{Ker}(\widetilde{SL}(E_{\mathbb{Z}}) \twoheadrightarrow SL(E_{\mathbb{Z}}))$ is the homomorphism $\mathbb{Z} \to \mathbb{Z}$ given by *multiplication by* 12, i.e.,

$$\chi\left(\widetilde{\tau}^{\measuredangle}\right) = 12, \quad \chi\left(\widetilde{\tau}^{|\measuredangle|}\right) = 6$$

(cf. the final portion of Sect. 2).

Finally, we recall that in Bogomolov's proof, one considers a *family of elliptic curves* (i.e., one-dimensional complex tori)

$$X \to S \quad (\subseteq \overline{S})$$

over a *hyperbolic Riemann surface* S of finite type (g, r) (so $2g - 2 + r > 0$) that has *stable bad reduction* at every *point at infinity* (i.e., point $\in \overline{S} \backslash S$) of some compact Riemann surface $\overline{S}$ that *compactifies* S. Such a family determines a *classifying morphism* $S \to \mathcal{M}$. The above discussion is summarized in the commutative diagrams and exact sequences of Figs. 2 and 3.

4 Estimates of displacements subject to indeterminacies

We conclude our review of Bogomolov's proof by briefly recalling the key points of the argument applied in this proof. These key points revolve around *estimates of displacements* that are subject to certain *indeterminacies*.

Write

$$\mathrm{Aut}_{\pi}\,(\mathbb{R}_{\geq 0})$$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{SL}(E) & \longrightarrow & SL(E) & \longrightarrow & 1 \\
 & & \bigcup & & \bigcup & & \bigcup & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{SL}(E_{\mathbb{Z}}) & \longrightarrow & SL(E_{\mathbb{Z}}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \chi & & \downarrow & & \\
1 & \longrightarrow & 12 \cdot \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/12 \cdot \mathbb{Z} & \longrightarrow & 1
\end{array}$$

Fig. 2 Exact sequences related to Bogomolov's proof

$$\begin{array}{ccccccccccc}
 & & \mathbb{Z} & = & \mathbb{Z} & & & \xrightarrow{\sim} & & 12 \cdot \mathbb{Z} & \\
 & & \bigcap & & \bigcap & & & & & \bigcap & \\
 & & \widetilde{SL}(E) & \supseteq & \widetilde{SL}(E_{\mathbb{Z}}) & \curvearrowright & & \overset{\chi}{\twoheadrightarrow} & & \mathbb{Z} & \curvearrowright \\
 & & \downarrow & & \downarrow & & (\omega^{\times}_{\mathcal{M}})^{\sim} & = & (\omega^{\times\otimes 12}_{\mathcal{M}})^{\sim} & \twoheadrightarrow & \mathbb{C} \\
 & & SL(E) & \supseteq & SL(E_{\mathbb{Z}}) & \curvearrowright & \downarrow & & \downarrow & & \downarrow \exp \\
 & & & & & & \omega^{\times}_{\widetilde{\mathcal{M}}} & \twoheadrightarrow & \omega^{\times\otimes 12}_{\widetilde{\mathcal{M}}} & \twoheadrightarrow & \mathbb{C}^{\times} \\
 & & & & & & \downarrow & & \downarrow & & \| \\
X & \longrightarrow & S & \longrightarrow & \mathcal{M} & \longleftarrow & \omega^{\times}_{\mathcal{M}} & \twoheadrightarrow & \omega^{\times\otimes 12}_{\mathcal{M}} & \twoheadrightarrow & \mathbb{C}^{\times}
\end{array}$$

Fig. 3 Fundamental groups related to Bogomolov's proof for the group of *self-homeomorphisms* $\mathbb{R}_{\geq 0} \xrightarrow{\sim} \mathbb{R}_{\geq 0}$ that *stabilize* and *restrict to the identity* on the subset $\pi \cdot \mathbb{N} \subseteq \mathbb{R}_{\geq 0}$ and $\mathbb{R}_{|\pi|}$ for the set of $\mathrm{Aut}_\pi(\mathbb{R}_{\geq 0})$-*orbits* of $\mathbb{R}_{\geq 0}$ [relative to the natural action of $\mathrm{Aut}_\pi(\mathbb{R}_{\geq 0})$ on $\mathbb{R}_{\geq 0}$]. Thus, one verifies easily that

$$\mathbb{R}_{|\pi|} = \left(\bigcup_{n \in \mathbb{N}} \{[n \cdot \pi]\} \right) \bigcup \left(\bigcup_{m \in \mathbb{N}} \{[(m \cdot \pi, (m+1) \cdot \pi)]\} \right)$$

—where we use the notation "$[-]$" to denote the element in $\mathbb{R}_{|\pi|}$ determined by an element or non-empty subset of $\mathbb{R}_{\geq 0}$ that lies in a single $\mathrm{Aut}_\pi(\mathbb{R}_{\geq 0})$-orbit. In particular, we observe that the natural order relation on $\mathbb{R}_{\geq 0}$ induces a *natural order relation* on $\mathbb{R}_{|\pi|}$.

For $\widetilde{\zeta} \in \widetilde{SL}(E)$, write

$$\delta\left(\widetilde{\zeta}\right) \stackrel{\text{def}}{=} \left\{ \left[\left| \widetilde{\zeta}(e) - e \right| \right] \mid e \in \left(E^{\angle} \right)^{\sim} \right\} \subseteq \mathbb{R}_{|\pi|}$$

—where the *absolute value of differences of elements of* $(E^{\angle})^{\sim}$ is computed with respect to some *fixed choice* of a homeomorphism $(E^{\angle})^{\sim} \xrightarrow{\sim} \mathbb{R}$ that belongs to the *natural* $\mathrm{Aut}_\pi(\mathbb{R})$-*orbit of homeomorphisms* discussed in Sect. 2, and we observe that it follows immediately from the *definition of* $\mathbb{R}_{|\pi|}$ that the subset $\delta(\widetilde{\zeta}) \subseteq \mathbb{R}_{|\pi|}$ is in fact *independent* of this fixed choice of homeomorphism.

Since (one verifies easily, from the *connectedness* of the Lie group $\widetilde{SL}(E)$, that) $\widetilde{\tau}^{|\measuredangle|}$ belongs to the *center* of the group $\widetilde{SL}(E)$, it follows immediately [from the *definition* of $\mathbb{R}_{|\pi|}$, by considering *translates* of $e \in (E^{\angle})^{\sim}$ by iterates of $\widetilde{\tau}^{|\measuredangle|}$] that the set $\delta(\widetilde{\zeta})$ is *finite*, hence admits a *maximal element*

$$\delta^{\sup}\left(\widetilde{\zeta}\right) \stackrel{\text{def}}{=} \sup\left(\delta\left(\widetilde{\zeta}\right)\right)$$

(cf. the *length* $\ell(-)$ of the discussion preceding [10, Lemma 3.7]). Thus,

$$\delta\left(\left(\widetilde{\tau}^{|\measuredangle|}\right)^n\right) = \{[|n| \cdot \pi]\}, \quad \delta^{\sup}\left(\left(\widetilde{\tau}^{|\measuredangle|}\right)^n\right) = [|n| \cdot \pi]$$

for $n \in \mathbb{Z}$ (cf. the discussion preceding [10, Lemma 3.7]). We shall say that $\widetilde{\zeta} \in \widetilde{SL}(E)$ is **minimal** if $\delta^{\sup}(\widetilde{\zeta})$ determines a *minimal element* of the set $\{\delta^{\sup}(\widetilde{\zeta} \cdot (\widetilde{\tau}^{\measuredangle})^n)\}_{n \in \mathbb{Z}}$.

Next, observe that the **cusp** "∞" discussed in Sect. 2 may be thought of as a *choice* of some rank one submodule $E_\infty \subseteq E_\mathbb{Z}$ for which there exists a rank one submodule $E_0 \subseteq E_\mathbb{Z}$—which may be thought of as a **cusp** "0"—such that the resulting natural inclusions determine an isomorphism

$$E_\infty \oplus E_0 \xrightarrow{\sim} E_\mathbb{Z}$$

of $\mathbb{Z}$-modules. Note that since E_∞ and E_0 are *free* $\mathbb{Z}$-*modules of rank one*, it follows (from the fact that the automorphism group of the group $\mathbb{Z}$ is of order two!) that there exist natural isomorphisms $E_\infty^{\otimes 2} \xrightarrow{\sim} E_0^{\otimes 2} \xrightarrow{\sim} \mathbb{Z}$. On the other hand, the *natural symplectic form* $\langle -, - \rangle_{E_\mathbb{Z}} \stackrel{\text{def}}{=} \langle -, - \rangle_E|_{E_\mathbb{Z}}$ on $E_\mathbb{Z}$ determines an isomorphism of E_∞ with the *dual* of E_0, hence (by applying the natural isomorphism $E_0^{\otimes 2} \xrightarrow{\sim} \mathbb{Z}$) a *natural isomorphism* $E_\infty \xrightarrow{\sim} E_0$.

This natural isomorphism $E_\infty \xrightarrow{\sim} E_0$ determines a *non-trivial unipotent automorphism* $\tau_\infty \in SL(E_\mathbb{Z})$ of $E_\mathbb{Z} = E_\infty \oplus E_0$ that fixes $E_\infty \subseteq E_\mathbb{Z}$—i.e., which may be thought of, relative to natural isomorphism $E_\infty \xrightarrow{\sim} E_0$, as the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$—as well as an $SL(E_\mathbb{Z})$-*conjugate unipotent automorphism* $\tau_0 \in SL(E_\mathbb{Z})$—i.e., which may be thought of, relative to natural isomorphism $E_\infty \xrightarrow{\sim} E_0$, as the matrix $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. Thus, the product

$$\tau_\infty \cdot \tau_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \in SL(E_\mathbb{Z})$$

lifts, relative to a suitable homeomorphism $(E^{\angle})^{\sim} \xrightarrow{\sim} \mathbb{R}$ that belongs to the *natural* $\mathrm{Aut}_{\pi}(\mathbb{R})$*-orbit of homeomorphisms* discussed in Sect. 2, to an element $\widetilde{\tau}_{\theta} \in \widetilde{SL}(E_{\mathbb{Z}})$ that induces the automorphism of $\mathbb{R}$ given by **translation** by θ for some $\theta \in \mathbb{R}$ such that $|\theta| = \frac{1}{3}\pi$.

The **key observations** that underlie Bogomolov's proof may be summarized as follows (cf. [10, Lemmas 3.6, 3.7]):

(B1) Every **unipotent** element $\tau \in SL(E)$ *lifts uniquely* to an element

$$\widetilde{\tau} \in \widetilde{SL}(E)$$

that *stabilizes* and *restricts to the identity* on some $(\widetilde{\tau}^{|\angle|})^{\mathbb{Z}}$**-orbit** of $(E^{\angle})^{\sim}$. Such a $\widetilde{\tau}$ is **minimal** and satisfies

$$\delta^{\sup}(\widetilde{\tau}) < [\pi].$$

(B2) Every **commutator** $[\widetilde{\alpha}, \widetilde{\beta}] \in \widetilde{SL}(E)$ of elements $\widetilde{\alpha}, \widetilde{\beta} \in \widetilde{SL}(E)$ satisfies

$$\delta^{\sup}\left(\left[\widetilde{\alpha}, \widetilde{\beta}\right]\right) < [2\pi].$$

(B3) Let $\widetilde{\tau}_{\infty}, \widetilde{\tau}_0 \in \widetilde{SL}(E_{\mathbb{Z}})$ be liftings of $\tau_{\infty}, \tau_0 \in SL(E_{\mathbb{Z}})$ as in (B1). Then

$$\widetilde{\tau}_{\infty} \cdot \widetilde{\tau}_0 = \widetilde{\tau}_{\theta}, \quad \text{and} \quad \theta = \tfrac{1}{3}\pi > 0.$$

In particular,

$$(\widetilde{\tau}_{\infty} \cdot \widetilde{\tau}_0)^3 = \widetilde{\tau}^{|\measuredangle|}, \quad \chi(\widetilde{\tau}_{\infty}) = \chi(\widetilde{\tau}_0) = 1, \chi\left(\widetilde{\tau}^{\measuredangle}\right) = 2 \cdot \chi(\widetilde{\tau}^{|\measuredangle|}) = 12.$$

Observation (*B1*) follows immediately, in light of the various definitions involved, together with the fact that $\widetilde{\tau}^{|\measuredangle|}$ belongs to the *center* of the group $\widetilde{SL}(E)$, from the fact τ *fixes* the [*distinct*!] images in $E^{\angle}$ of $\pm v \in E$ for some *nonzero* $v \in E$.

Next, let us write $|SL(E)| \stackrel{\mathrm{def}}{=} SL(E)/\{\pm 1\}$. Then observe that since the generator $\widetilde{\tau}^{|\measuredangle|}$ of $\mathrm{Ker}(\widetilde{SL}(E) \twoheadrightarrow SL(E) \twoheadrightarrow |SL(E)|)$ belongs to the *center* of $\widetilde{SL}(E)$, it follows that every commutator $[\widetilde{\alpha}, \widetilde{\beta}]$ as in *observation* (*B2*) is *completely determined* by the respective images $|\alpha|, |\beta| \in |SL(E)|$ of $\widetilde{\alpha}, \widetilde{\beta} \in \widetilde{SL}(E)$. Now recall (cf. the proof of Lemma 3.5 [10]) that it follows immediately from an *elementary linear algebra* argument—i.e., consideration of a *solution* "x" to the equation

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = 0$$

associated to an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ such that $c \neq 0$—that every element of $SL(E)$ other than $-1 \in SL(E)$ may be written as a *product of two unipotent elements* of $SL(E)$. In particular, we conclude that every commutator $[\widetilde{\alpha}, \widetilde{\beta}] = (\widetilde{\alpha} \cdot \widetilde{\beta} \cdot \widetilde{\alpha}^{-1}) \cdot \widetilde{\beta}^{-1}$ as in observation (B2) may be written as a *product*

$$\widetilde{\tau}_1 \cdot \widetilde{\tau}_2 \cdot \widetilde{\tau}_2^* \cdot \widetilde{\tau}_1^*$$

of *four minimal liftings* "$\widetilde{\tau}$" as in (B1) such that $\widetilde{\tau}_1^*, \widetilde{\tau}_2^*$ are $\widetilde{SL}(E)$*-conjugate* to $\widetilde{\tau}_1^{-1}, \widetilde{\tau}_2^{-1}$, respectively. On the other hand, it follows immediately from the fact that the action on $E^{\angle}$ of any *non-trivial* (i.e., $\neq 1$) *unipotent element* of $SL(E)$ has *precisely two fixed points* (i.e., precisely one $\{\pm 1\}$-orbit of fixed points) that, for $i = 1, 2$, there exists an element $\epsilon_i \in \{\pm 1\}$ such that, relative to the action of $\widetilde{SL}(E)$ on $(E^{\angle})^{\sim} \xrightarrow{\sim} \mathbb{R}$, $\widetilde{\tau}_i^{\epsilon_i}$ maps every element $x \in \mathbb{R}$ to an element $\mathbb{R} \ni \widetilde{\tau}_i^{\epsilon_i}(x) \geq x$. [Indeed, consider the *continuity* properties of the map $\mathbb{R} \ni x \mapsto \widetilde{\tau}_i(x) - x \in \mathbb{R}$, which is *invariant* with respect to *translation* by π in its domain!] Moreover, since any element of $\widetilde{SL}(E)$ induces a self-homeomorphism of $(E^{\angle})^{\sim} \xrightarrow{\sim} \mathbb{R}$ that *commutes* with the action of $\widetilde{\tau}^{|\measuredangle|}$, hence is necessarily *strictly monotone increasing*, we conclude that, for $i = 1, 2$, $(\widetilde{\tau}_i^*)^{\epsilon_i}$ maps every element $x \in \mathbb{R}$ to an element $\mathbb{R} \ni (\widetilde{\tau}_i^*)^{\epsilon_i}(x) \leq x$. In particular, any computation of the *displacements* $\in \mathbb{R}$ that occur as the result of applying the above product $\widetilde{\tau}_1 \cdot \widetilde{\tau}_2 \cdot \widetilde{\tau}_2^* \cdot \widetilde{\tau}_1^*$ to some element of $(E^{\angle})^{\sim} \xrightarrow{\sim} \mathbb{R}$ yields, in light of the estimates $\delta^{\sup}(\widetilde{\tau}_1) = \delta^{\sup}(\widetilde{\tau}_1^*) < [\pi]$, $\delta^{\sup}(\widetilde{\tau}_2) = \delta^{\sup}(\widetilde{\tau}_2^*) < [\pi]$ of (B1), a sum

$$\left(\left((a_1^* + a_2^*) + a_2\right) + a_1\right) = (a_1 + a_1^*) + (a_2 + a_2^*) \in \mathbb{R}$$

for suitable elements

$$a_1 \in \epsilon_1 \cdot [0, \pi) \subseteq \mathbb{R}; \quad a_1^* \in -\epsilon_1 \cdot [0, \pi) \subseteq \mathbb{R};$$
$$a_2 \in \epsilon_2 \cdot [0, \pi) \subseteq \mathbb{R}; \quad a_2^* \in -\epsilon_2 \cdot [0, \pi) \subseteq re.$$

Thus, the estimate $\delta^{\sup}([\widetilde{\alpha}, \widetilde{\beta}]) < [2\pi]$ of observation (B2) follows immediately from the estimates $|a_1 + a_1^*| < \pi$, $|a_2 + a_2^*| < \pi$.

Next, observe that since $\pi < 2\pi - \frac{1}{3}\pi$, it follows immediately that $\{[0], [(0, \pi)]\} \cap \delta(\widetilde{\tau}_\theta \cdot (\widetilde{\tau}^{\measuredangle})^n) = \emptyset$ for $n \neq 0$. On the other hand, (B1) implies that $[0] \in \delta(\widetilde{\tau}_0)$ and $\delta^{\sup}(\widetilde{\tau}_\infty) < [\pi]$, and hence that $\{[0], [(0, \pi)]\} \cap \delta(\widetilde{\tau}_\infty \cdot \widetilde{\tau}_0) \neq \emptyset$. Thus, the relation $\widetilde{\tau}_\infty \cdot \widetilde{\tau}_0 = \widetilde{\tau}_\theta$ of *observation (B3)* follows immediately; the *positivity* of θ follows immediately from the *clockwise* nature (cf. the definition "$\widetilde{\tau}^{\measuredangle}$" in the final portion of Sect. 2) of the assignments $\binom{1}{0} \mapsto \binom{0}{-1}$, $\binom{0}{1} \mapsto \binom{1}{1}$ determined by $\tau_\infty \cdot \tau_0$.

Next, recall the well-known *presentation* via generators $\alpha_1^S, \ldots, \alpha_g^S, \beta_1^S, \ldots, \beta_g^S, \gamma_1^S, \ldots, \gamma_r^S$ (where $\gamma_1^S, \ldots, \gamma_r^S$ generate the respective *inertia groups* at the *points at infinity* $\overline{S} \backslash S$ of S) subject to the *relation*

$$\left[\alpha_1^S, \beta_1^S\right] \cdot \ldots \cdot \left[\alpha_g^S, \beta_g^S\right] \cdot \gamma_1^S \cdot \ldots \cdot \gamma_r^S = 1$$

of the fundamental group Π_S of the Riemann surface S. These generators map, via the outer homomorphism $\Pi_S \to \Pi_{\mathcal{M}}$ induced by the *classifying morphism* of the *family of elliptic curves* under consideration, to elements $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_r$ subject to the *relation*

$$[\alpha_1, \beta_1] \cdot \ldots \cdot \left[\alpha_g, \beta_g\right] \cdot \gamma_1 \cdot \ldots \cdot \gamma_r = 1$$

of the fundamental group $\Pi_{\mathcal{M}} = SL(E_{\mathbb{Z}})$ (for a suitable choice of basepoint) of $\mathcal{M}$.

Next, let us choose *liftings* $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_g, \widetilde{\beta}_1, \ldots, \widetilde{\beta}_g, \widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_r$ of $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_r$ to elements of $\widetilde{SL}(E_{\mathbb{Z}})$ such that $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_r$ are **minimal** liftings as in (B1). Thus, we obtain a *relation*

$$\left[\widetilde{\alpha}_1, \widetilde{\beta}_1\right] \cdot \ldots \cdot \left[\widetilde{\alpha}_g, \widetilde{\beta}_g\right] \cdot \widetilde{\gamma}_1 \cdot \ldots \cdot \widetilde{\gamma}_r = \left(\widetilde{\tau}^{\measuredangle}\right)^{n^{\measuredangle}} = \left(\widetilde{\tau}^{|\measuredangle|}\right)^{2n^{\measuredangle}}$$

in $\widetilde{SL}(E_{\mathbb{Z}})$ for some $n^{\measuredangle} \in \mathbb{Z}$. The situation under consideration is summarized in Fig. 4.

Now it follows from the various definitions involved, together with the well-known theory of *Tate curves*, that, for $i = 1, \ldots, r$,

the element γ_i is an $SL(E_{\mathbb{Z}})$-conjugate of $\tau_\infty^{\nu_i}$

for some $\nu_i \in \mathbb{N}$. Put another way, ν_i is the *order of the q-parameter* of the *Tate curve* determined by the given family $X \to S$ at the point at infinity corresponding to γ_i^S.

$$\begin{array}{ccccccc}
 & & \Pi_S & \longrightarrow & \Pi_{\mathcal{M}} & & \\
 & & & & \| & & \\
\widetilde{SL}(E) & \supseteq & \widetilde{SL}(E_{\mathbb{Z}}) & \twoheadrightarrow & SL(E_{\mathbb{Z}}) & \subseteq & SL(E) \\
\curvearrowright & & & & & & \curvearrowright \\
(E^{\angle})^{\sim} & & & \twoheadrightarrow & & & E^{\angle}
\end{array}$$

Fig. 4 The setup of Bogomolov's proof

Thus, by applying $\chi(-)$ to the above *relation*, we conclude (cf. the discussion preceding [10, Lemma 3.7]) from the *equalities* in the final portion of (B3) (together with the evident fact that *commutators* necessarily lie in the *kernel* of $\chi(-)$) that

(B4) The "orders of q-parameters" $v_1, \ldots, v_r$ satisfy the **equality**

$$\sum_{i=1}^{r} v_i = 12n^{\measuredangle}$$

—where $n^{\measuredangle} \in \mathbb{Z}$ is the quantity defined in the above discussion.

On the other hand, by applying $\delta^{\sup}(-)$ to the above *relation*, we conclude (cf. the discussion following the proof of [10, Lemma 3.7]) from the *estimates* of (B1) and (B2), the *equality* of (B4), and the equality $\delta^{\sup}((\widetilde{\tau}^{|\angle|})^n) = [\ |n| \cdot \pi\ ]$, for $n \in \mathbb{Z}$, that

$$\left(\sum_{i=1}^{r} \pi\right) + \left(\sum_{j=1}^{g} 2\pi\right) > 2\pi \cdot n^{\measuredangle} = \frac{1}{6} \cdot \pi \cdot \sum_{i=1}^{r} v_i$$

—i.e., that

(B5) The "orders of q-parameters" $v_1, \ldots, v_r$ satisfy the **estimate**

$$\frac{1}{6} \cdot \sum_{i=1}^{r} v_i < 2g + r$$

—where (g, r) is the type of the hyperbolic Riemann surface S.

Finally, we conclude (cf. the discussion following the proof of [10, Lemma 3.7]) the **geometric** version of the **Szpiro inequality**

$$\frac{1}{6} \cdot \sum_{i=1}^{r} v_i \leq 2g - 2 + r$$

by applying (B5) (multiplied by a *normalization factor* $\frac{1}{d}$) to the families obtained from the given family $X \to S$ by base-changing to *finite étale Galois coverings* of S of *degree* d and *passing to the limit* $d \to \infty$.

5 Similarities between the two theories

We are now in a position to *reap the benefits* of the formulation of Bogomolov's proof given above, which is *much closer "culturally"* to inter-universal Teichmüller theory than the formulation of [1,10].

We begin by considering the relationship between Bogomolov's proof and (IU1), i.e., the theory of $\Theta^{\pm \mathrm{ell}}$**NF-Hodge theaters**, as developed in [6]. First of all, Bogomolov's proof clearly centers around the **hyperbolic geometry** of the **upper half-plane**. This aspect of Bogomolov's proof is directly reminiscent of the detailed analogy discussed in [6, Remark 6.12.3]; [6, Fig. 6.4], between the structure of $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters and the classical geometry of the upper half-plane—cf., e.g., the discussion of the *natural identification* of $E^{|\angle|}$ with the boundary $\partial \widetilde{\mathcal{M}}$ of $\widetilde{\mathcal{M}}$ in Sect. 2; the discussion of the *boundary of the upper half-plane* in [6], Remark 6.12.3, (iii). In particular, one may think of

> the **additive** $\mathbb{F}_l^{\rtimes \pm}$**-symmetry** portion of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater as corresponding to the **unipotent** transformations τ_∞, τ_0, γ_i

that appear in Bogomolov's proof and of

> the **multiplicative** $\mathbb{F}_l^{\circledast}$**-symmetry** portion of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater as corresponding to the **toral**/"**typically non-unipotent**" transformations $\tau_\infty \cdot \tau_0$, α_i, β_i

that appear in Bogomolov's proof, i.e., typically as *products of two non-commuting unipotent transformations* (cf. the proof of (B2)!). Here, we recall that the notation $\mathbb{F}_l^{\rtimes \pm}$ denotes the semi-direct product group $\mathbb{F}_l \rtimes \{\pm 1\}$ (relative to the natural action of $\{\pm 1\}$ on the underlying additive group of $\mathbb{F}_l$), while the notation $\mathbb{F}_l^{\circledast}$ denotes the quotient of the multiplicative group $\mathbb{F}_l^{\times}$ by the action of $\{\pm 1\}$.

One central aspect of the theory of $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters developed in [6] lies in the goal of somehow **"simulating"** a situation in which the module of l-torsion points of the given elliptic curve over a number field admits a **"global multiplicative subspace"** (cf. the discussion of [6, §I1]; [6, Remark 4.3.1]). One way to understand this sort of "simulated" situation is in terms of the **one-dimensional additive geometry** associated to a nontrivial unipotent transformation. That is to say, whereas, from an *a priori* point of view, the one-dimensional additive geometries associated to *conjugate, non-commuting unipotent transformations* are **distinct** and **incompatible**, the "simulation" under consideration may be understood as consisting of the establishment of some sort of **geometry** in which these distinct, incompatible one-dimensional additive geometries are somehow **"identified"** with one another as a **single, unified one-dimensional additive geometry**. This fundamental aspect of the theory of $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters in [6] is thus reminiscent of the

> **single, unified one-dimensional objects** $E^{\angle} \left(\stackrel{\sim}{\to} \mathbb{S}^1 \right), \ \left(E^{\angle}\right)^{\sim} \left(\stackrel{\sim}{\to} \mathbb{R} \right)$

in Bogomolov's proof which admit natural actions by *conjugate, non-commuting unipotent transformations* $\in SL(E)$ (i.e., such as τ_∞, τ_0) and their *minimal* liftings to $\widetilde{SL}(E)$ [i.e., such as $\widetilde{\tau}_\infty$, $\widetilde{\tau}_0$—cf. (B1)].

The issue of simulation of a "global multiplicative subspace" as discussed in [6, Remark 4.3.1] is closely related to the application of **absolute anabelian geometry** as developed in [5], i.e., to the issue of establishing global arithmetic analogues for number fields of the classical theories of **analytic continuation** and **Kähler metrics**, constructed via the use of **logarithms**, on hyperbolic Riemann surfaces (cf. [6, Remarks 4.3.2, 4.3.3, 5.1.4]). These aspects of inter-universal Teichmüller theory are, in turn, closely related (cf. the discussion of [6, Remark 4.3.3]) to the application in [8] of the theory of **log-shells** [cf. (IU3)] as developed in [5] to the task of constructing **multiradial mono-analytic containers**, as discussed in the Introductions to [8,9]. These multiradial mono-analytic containers play the *crucial role* of furnishing containers for the various objects of interest—i.e., the **theta value** and **global number field** portions of Θ-*pilot objects*—that, although subject to various **indeterminacies** (cf. the discussion of the indeterminacies (Ind1), (Ind2), (Ind3) in the Introduction to [8]), allow one to obtain the **estimates** (cf. [8, Remark 3.10.1, (iii)]) of these objects of interest as discussed in detail in [9, §1, §2] (cf., especially, the proof of [9, Theorem 1.10]). These aspects of inter-universal Teichmüller theory may be thought of as corresponding to the essential use of $(E^{\angle})^{\sim}$ ($\xrightarrow{\sim}$ $\mathbb{R}$) in Bogomolov's proof, i.e., which is reminiscent of the **log-shells** that appear in inter-universal Teichmüller theory in many respects:

(L1) The object $(\omega^{\measuredangle}_{\mathcal{M}})^{\sim}$ that appears in Bogomolov's proof may be thought of as corresponding to the **holomorphic log-shells** of inter-universal Teichmüller theory, i.e., in the sense that it may be thought of as a sort of **"logarithm"** of the **"holomorphic family of copies of the group of units** $\mathbb{S}^1$" constituted by $\omega^{\measuredangle}_{\widetilde{\mathcal{M}}}$—cf. the discussion of variation of **complex structure** in Sect. 2.

(L2) Each fiber over $\widetilde{\mathcal{M}}$ of the "holomorphic log-shell" $(\omega^{\measuredangle}_{\mathcal{M}})^{\sim}$ maps isomorphically (cf. Fig. 1) to $(E^{\angle})^{\sim}$, an essentially **real analytic** object that is **independent** of the varying complex structures discussed in (L1), hence may be thought of as corresponding to the **mono-analytic log-shells** of inter-universal Teichmüller theory.

(L3) Just as in the case of the mono-analytic log-shells of inter-universal Teichmüller theory (cf., especially, the proof of [9, Theorem 1.10]), $(E^{\angle})^{\sim}$ serves as a **container** for **estimating** the various objects of interest in Bogomolov's proof, as discussed in (B1), (B2), objects which are subject to the **indeterminacies** constituted by the action of $\mathrm{Aut}_{\pi}(\mathbb{R})$, $\mathrm{Aut}_{\pi}(\mathbb{R}_{\geq 0})$ [cf. the indeterminacies (Ind1), (Ind2), (Ind3) in inter-universal Teichmüller theory].

(L4) In the context of the estimates of (L3), the estimates of **unipotent** transformations given in (B1) may be thought of as corresponding to the estimates involving **theta values** in inter-universal Teichmüller theory, while the estimates of **"typically non-unipotent"** transformations given in (B2) may be thought of as corresponding to the estimates involving **global number field** portions of Θ-pilot objects in inter-universal Teichmüller theory.

(L5) As discussed in the [6, §I1], the Kummer theory surrounding the **theta values** is closely related to the **additive symmetry** portion of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater, i.e., in which **global synchronization** of $\pm$-**indeterminacies** (cf. [6, Remark 6.12.4, (iii)]) plays a fundamental role. Moreover, as discussed in [8, Remark 2.3.3, (vi), (vii), (viii)], the essentially **local** nature of the **cyclotomic rigidity isomorphisms** that appear in the Kummer theory surrounding the theta values renders them free of any $\pm$-**indeterminacies**. These phenomena of **rigidity** with respect to $\pm$-indeterminacies in inter-universal Teichmüller theory are highly reminiscent of the *crucial estimate* of (B1) involving

the **volume** π of a **fundamental domain** $\overline{D}^{\angle}$

for the action of $\{\pm 1\}$ on $E^{\angle}$ (i.e., as opposed to the volume 2π of the $\{\pm 1$-orbit $\pm\overline{D}^{\angle}$ of $\overline{D}^{\angle}$!), as well as of the **uniqueness** of the **minimal** liftings of (B1). In this context, we also recall that the *additive symmetry* portion of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater, which depends, in an essential way, on the *global synchronization* of $\pm$-*indeterminacies* (cf. [6, Remark 6.12.4, (iii)]), is used in inter-universal Teichmüller theory to establish **conjugate synchronization**, which plays an indispensable role in the construction of **bi-coric mono-analytic log-shells** (cf. [8, Remark 1.5.1]). This state of affairs is highly reminiscent of the important role played by $E^{\angle}$, as opposed to $E^{|\angle|} = E^{\angle}/\{\pm 1\}$, in Bogomolov's proof.

(L6) As discussed in the [6, §I1], the Kummer theory surrounding the **number fields** under consideration is closely related to the **multiplicative symmetry** portion of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater, i.e., in which one always works with **quotients** via the action of ± 1. Moreover, as discussed in [8, Remark 2.3.3, (vi), (vii), (viii)] (cf. also [7, Remark 4.7.3, (i)]), the essentially **global** nature—which necessarily involves at least **two localizations**, corresponding to a **valuation** [say, "0"] and the corresponding **inverse valuation** [i.e., "∞"] of a function field—of the **cyclotomic rigidity isomorphisms** that appear in the Kummer theory surrounding number fields causes them to be **subject** to $\pm$-**indeterminacies**. These $\pm$-*indeterminacy* phenomena in inter-universal Teichmüller theory are highly reminiscent of the *crucial estimate* of (B2)—which arises from considering products of **two non-commuting unipotent transformations**, i.e., corresponding to "two distinct localizations"—involving

the **volume** 2π of the $\{\pm 1\}$-**orbit** $\pm\overline{D}^{\angle}$ of a **fundamental domain** $\overline{D}^{\angle}$

for the action of $\{\pm 1\}$ on $E^{\angle}$ (i.e., as opposed to the volume π of $\overline{D}^{\angle}$!).

(L7) The **analytic continuation** aspect (say, from "∞" to "0") of inter-universal Teichmüller theory–i.e., via the technique of **Belyi cuspidalization** as discussed in [6, Remarks 4.3.2, 5.1.4]—may be thought of as corresponding to the "analytic continuation" inherent in the **holomorphic** structure of the "holomorphic log-shell $(\omega^{\angle}_{\mathcal{M}})^{\sim}$," which relates, in particular, the localizations at the cusps "∞" and "0."

Here, we note in passing that one way to understand certain aspects of the phenomena discussed in (L4)–(L6) is in terms of the following *"general principle"*: Let k be an algebraically closed field. Write $k^{\times}$ for the multiplicative group of nonzero elements of k, $PGL_2(k) \stackrel{\mathrm{def}}{=} GL_2(k)/k^{\times}$. Thus, by thinking in terms of fractional linear transformations, one may regard $PGL_2(k)$ as the group of k-automorphisms of the projective line $P \stackrel{\mathrm{def}}{=} \mathbb{P}^1_k$ over k. We shall say that an element of $PGL_2(k)$ is *unipotent* if it arises from a unipotent element of $GL_2(k)$. Let $\xi \in PGL_2(k)$ be a non-trivial element. Write P^{ξ} for the set of k-rational points of P that are **fixed** by ξ. Then observe that

ξ is **unipotent** $\iff$ P^{ξ} is of **cardinality one**;

ξ is **non-unipotent** $\iff$ P^{ξ} is of **cardinality two**.

That is to say,

General principle:

- A non-trivial **unipotent** element $\xi \in PGL_2(k)$ may be regarded as expressing a **local geometry**, i.e., the geometry in the neighborhood of a **single point** [namely the unique fixed point of ξ]. Such a "local geometry"—that is to say, more precisely, the set P^{ξ} of *cardinality one*—does *not* admit a *reflection, or* $\pm$-, *symmetry.*
- By contrast, a non-trivial **non-unipotent** element $\xi \in PGL_2(k)$ may be regarded as expressing a **global geometry**, i.e., the "toral" geometry corresponding to a **pair of**

> **points** "0" and "∞" [namely the two fixed points of ξ]. Such a "global toral geometry"—that is to say, more precisely, the set P^{ξ} of *cardinality two*—typically *does* admit a *"reflection, or* $\pm$*-, symmetry"* (i.e., which permutes the two points of P^{ξ}).

Next, we recall that the suitability of the multiradial mono-analytic containers furnished by **log-shells** for **explicit estimates** (cf. [8, Remark 3.10.1, (iii)]) lies in sharp contrast to the **precise**, albeit somewhat **tautological**, nature of the correspondence [cf. (IU2)] concerning **arithmetic degrees** of objects of interest (i.e., *q-pilot* and Θ*-pilot* objects) given by the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link** (cf. [8, Definition 3.8, (i), (ii)]; [8, Remark 3.10.1, (ii)]). This precise correspondence is reminiscent of the **precise**, but relatively **"superficial"** [i.e., by comparison with the estimates (B1), (B2)], relationships concerning degrees [cf. (B4)] that arise from the homomorphism χ [i.e., which is denoted "deg" in [10]!]. On the other hand, the final estimate (B5) requires one to apply *both* the precise computation of (B4) *and* the non-trivial estimates of (B1), (B2). This state of affairs is highly reminiscent of the discussion surrounding [8, Fig. I.8], of **two equivalent ways** to compute log-volumes, i.e., the *precise correspondence* furnished by the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link** and the *non-trivial estimates* via the multiradial mono-analytic containers furnished by the **log-shells**.

Finally, we observe that the *complicated interplay* between **"Frobenius-like"** and **"étale-like"** objects in inter-universal Teichmüller theory may be thought of as corresponding to the complicated interplay in Bogomolov's proof between

> **complex holomorphic** objects such as the **holomorphic line bundle** $\omega_{\mathcal{M}}$ and the natural surjections $\omega^{\times}_{\mathcal{M}} \twoheadrightarrow \omega^{\times\otimes 12}_{\mathcal{M}} \twoheadrightarrow \mathbb{C}^{\times}$ arising from the **discriminant modular form**

—i.e., which correspond to *Frobenius-like* objects in inter-universal Teichmüller theory–and

> the **local system** $\mathcal{E}_{\mathcal{M}}$ and the various **fundamental groups** [and morphisms between such fundamental groups such as χ] that appear in Fig. 3

—i.e., which correspond to *étale-like* objects in inter-universal Teichmüller theory.

The analogies discussed above are summarized in Table 1.

6 Differences between the two theories

In a word, the most essential difference between inter-universal Teichmüller theory and Bogomolov's proof appears to lie in the

> **absence** *in Bogomolov's proof of*
>
> **Gaussian distributions** *and* **theta functions**,

i.e., which play a central role in inter-universal Teichmüller theory.

In some sense, Bogomolov's proof may be regarded as arising from the **geometry** surrounding the **natural symplectic form**

$$\langle -, - \rangle_E \stackrel{\mathrm{def}}{=} \langle -, - \rangle_{\mathcal{E}}|_E$$

on the **two-dimensional** $\mathbb{R}$**-vector space** E. The natural arithmetic analogue of this symplectic form is the **Weil pairing** on the **torsion points**—i.e., such as the l-torsion points that appear in inter-universal Teichmüller theory–of an elliptic curve over a number field.

Table 1 Similarities between the two theories

Inter-universal Teichmüller Theory	Bogomolov's proof
$\mathbb{F}_l^{\rtimes\pm}$-, $\mathbb{F}_l^{\circledast}$-**symmetries of** $\Theta^{\pm\mathrm{ell}}$***NF*-Hodge theaters**	**Unipotent, toral/non-unipotent symmetries of upper half-plane**
Simulation of global multiplicative subspace	$\widetilde{SL}(E) \curvearrowright SL(E) \curvearrowright (\mathbb{R} \overset{\sim}{\to})(E^{\angle})^{\sim} \twoheadrightarrow E^{\angle} (\overset{\sim}{\to} \mathbb{S}^1)$
Holomorphic log-shells, analytic continuation "$\infty \rightsquigarrow 0$"	**"Holomorphic family" of fibers of** $(\omega^{\measuredangle}_{\mathcal{M}})^{\sim} \to \widetilde{\mathcal{M}}$, **e.g., at "$\infty$," "0"**
Multiradial mono-analytic containers via log-shells subject to indeterminacies (Ind1), (Ind2), (Ind3)	**Real analytic** $\widetilde{SL}(E) \curvearrowright (E^{\angle})^{\sim} (\overset{\sim}{\to} \mathbb{R})$ **subject to indeterminacies via actions of** $\mathrm{Aut}_{\pi}(\mathbb{R})$, $\mathrm{Aut}_{\pi}(\mathbb{R}_{\geq 0})$
$\pm$-**Rigidity of "local" Kummer theory, cyclotomic rigidity surrounding theta values, conjugate synchronization**	Estimate (B1) via π of **unique minimal liftings of unipotent transformations,** $E^{\angle}$ **(as opposed to** $E^{\mid\angle\mid}$**!)**
$\pm$-**Indeterminacy of "global" Kummer theory, cyclotomic rigidity surrounding number fields**	Estimate (B2) via 2π of **commutators of products of two non-commuting unipotent transformations**
Arithmetic degree computations via **precise** $\Theta^{\times\mu}_{\mathrm{LGP}}$**-link versus log-shell estimates**	Degree computations via **precise homomorphism** χ **(B4) versus** δ^{sup} **estimates (B1), (B2)**
Frobenius-like versus étale-like objects	**Complex holomorphic objects such as line bundles versus local systems, fundamental groups**

On the other hand, one fundamental difference between this Weil pairing on torsion points and the symplectic form $\langle - , - \rangle_E$ is the following:

> Whereas the field $\mathbb{R}$ over which the symplectic form $\langle - , - \rangle_E$ is defined may be regarded as a **subfield**—i.e.,
>
> $$\exists\, \mathbb{R} \hookrightarrow \mathbb{C}$$
>
> —of the field of definition $\mathbb{C}$ of the algebraic schemes (or stacks) under consideration, the field $\mathbb{F}_l$ over which the Weil pairing on l-torsion points is defined **cannot** be regarded as a **subfield**—i.e.,
>
> $$\nexists\, \mathbb{F}_l \hookrightarrow \mathbb{Q}$$
>
> —of the number field over which the (algebraic) elliptic curve under consideration is defined.

This phenomenon of *compatibility/incompatibility* of *fields of definition* is reminiscent of the **"mysterious tensor products"** that occur in *p-adic Hodge theory*, i.e., in which the "$\mathbb{Z}_p$" that acts on a p-adic Tate module is *identified* (despite its somewhat alien nature!) with the "$\mathbb{Z}_p$" that includes as a subring of the structure sheaf of the p-adic scheme under consideration (cf. the discussion of [3, Remark 3.7]; the final portion of [4, Remark 2.16.2]; [6, Remarks 4.3.1, 4.3.2]; [6, Remark 6.12.3, (i), (ii)]; [9, Remark 3.3.2]). Here, we observe further that the *former* "$\mathbb{Z}_p$," as well as the fields of definition of the symplectic form $\langle - , - \rangle_E$ and the Weil pairing on torsion points, are, from the point of view of inter-universal Teichmüller theory, **étale -like** objects, whereas the *latter* "$\mathbb{Z}_p$," as well as other instances of subrings of the structure sheaf of the scheme under consideration, are

Frobenius-like objects. That is to say, the point of view of inter-universal Teichmüller theory may be summarized as follows:

> Certain *geometric* aspects—i.e., aspects that, in effect, correspond to the **geometry** of the classical **upper half-plane** (cf. [6, Remark 6.12.3])—of the *a priori* **incompatibility** of *fields of definition* in the case of **elliptic curves over number fields** are, in some sense, overcome in inter-universal Teichmüller theory by applying various **absolute anabelian algorithms** to pass from étale-like to Frobenius-like objects, as well as various **cyclotomic rigidity algorithms** to pass, via **Kummer theory**, from Frobenius-like to étale-like objects.

Indeed, as discussed in [6, Remarks 4.3.1, 4.3.2], it is precisely this circle of ideas that forms the *starting point* for the *construction of* $\Theta^{\pm \mathrm{ell}}$NF-*Hodge theaters* given in [6], by applying the *absolute anabelian geometry* of [5].

One way to understand the gap between *fields of definition* of first cohomology modules or modules of torsion points, on the one hand, and the *field of definition* of the given base scheme, on the other, is to think of elements of fields/rings of the *former* sort as objects that occur as **exponents** of regular functions on the base scheme, i.e., elements of rings that naturally contain fields/rings of the *latter* sort. For instance, this sort of situation may be seen at a very explicit level by consider the *powers of the q-parameter* that occur in the theory of Tate curves over p-adic fields (cf. the discussion of the final portion of [4, Remark 2.16.2]). From this point of view, the approach of inter-universal Teichmüller theory may be summarized as follows:

> Certain *function-theoretic* aspects of the *a priori* ***incompatibility*** of *fields of definition* in the case of **elliptic curves over number fields** are, in some sense, overcome in inter-universal Teichmüller theory by working with **Gaussian distributions** and **theta functions**, i.e., which may be regarded, in effect, as **exponentiations** of the symplectic form $\langle -, - \rangle_E$ that appears in Bogomolov's proof.

Indeed, it is precisely as a result of such *exponentiation* operations that one is obliged to work, in inter-universal Teichmüller theory, with **arbitrary iterates** of the $\mathfrak{log}$-link (cf. the theory of [5,8]; the discussion of [8, Remark 1.2.2]) in order to relate and indeed *identify*, in effect, the *function theory of exponentiated objects* with the *function theory of non-exponentiated objects*. This situation differs somewhat from the *single application of the logarithm* constituted by the covering $(E^{\angle})^{\sim} \twoheadrightarrow E^{\angle}$ in Bogomolov's proof.

So far in the present Sect. 6, our discussion has centered around

- the **geometry** of $\Theta^{\pm \mathrm{ell}}$NF-**Hodge theaters** (as discussed in [6, §4–§6]) and
- the **multiradial representation via mono-analytic log-shells** (cf. [8, Theorem 3.11, (i), (ii)])

of inter-universal Teichmüller theory, which correspond, respectively, to the **symplectic geometry of the upper half-plane** (cf. §1) and the $\delta^{\sup}$ **estimates** (cf. (B1), (B2)) of Bogomolov's proof.

On the other hand, the degree computations via the homomorphism χ, which arises, in essence, by considering the **discriminant modular form**, also play a key role [cf. (B4)] in Bogomolov's proof. One may think of this aspect of Bogomolov's proof as consisting of the application of the discriminant modular form to *relate* the symplectic geometry discussed in Sect. 2—cf., especially, the natural $SL(E)$-**torsor** structure on $\omega^{\angle}_{\widetilde{\mathcal{M}}}$—to the conventional **algebraic** theory of **line bundles** and **divisors** on the algebraic stack $\mathcal{M}$. In particular, this aspect of Bogomolov's proof is reminiscent of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-**link**, i.e., which serves to relate the *Gaussian distributions* (that is to say, *exponentiated symplectic forms*) that appear in the *multiradial representation via mono-analytic log-shells* to the conventional theory of **arithmetic line bundles** on the number field under consideration. We remark in passing that this state of affairs is reminiscent of the point of view discussed in [2, §1.2, §1.3.2], to the effect that the constructions of scheme-theoretic Hodge–Arakelov theory (i.e., which may be regarded as a sort of scheme-theoretic precursor of inter-universal Teichmüller theory) may be thought of as a sort of *function-theoretic vector bundle version* of the *discriminant modular form*. The $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link is **not compatible** with the various **ring/scheme structures**—i.e., the **"arithmetic holomorphic structures"**—in its domain and codomain. In order to surmount this incompatibility, one must avail oneself of the theory of **multiradiality** developed in [7,8]. The non-ring-theoretic nature of the resulting *multiradial representation via mono-analytic log-shells*—cf. [8, Theorem 3.11, (i), (ii)]; the discussion of **inter-universality** in [9, Remark 3.6.3, (i)]—of inter-universal Teichmüller theory may then be thought of as corresponding to the **real analytic** (i.e., non-holomorphic) nature of the **symplectic geometry** that appears in Bogomolov's proof. In this context, we recall that

(E1) one *central feature* of Bogomolov's proof is the following *fundamental difference* between the *crucial* **estimate** (B1), which arises from the (non-holomorphic) **symplectic geometry** portion of Bogomolov's proof, on the one hand, and the homomorphism χ, on the other: whereas, for integers $N \geq 1$, the homomorphism χ maps Nth *powers* of elements $\widetilde{\tau}$ as in (B1) to *multiples by* N of elements $\in \mathbb{Z}$, the estimate $\delta^{\sup}(-) < [\pi]$ of (B1) is **unaffected** when one replaces an element $\widetilde{\tau}$ by such an Nth *power* of $\widetilde{\tau}$.

This central feature of Bogomolov's proof is *highly reminiscent* of the situation in inter-universal Teichmüller theory in which

(E2) although the **multiradial representation** *of* Θ*-pilot objects via mono-analytic log-shells* in the domain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link is related, via the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, to q-pilot objects in the codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, the **same** multiradial representation of the **same** Θ-pilot objects may related, in precisely the *same* fashion, to **arbitrary N-th powers** of q-pilot objects, for integers $N \geq 2$

(cf. the discussion of [8, Remark 3.12.1, (ii)]).

Thus, in summary, if, relative to the point of view of Bogomolov's proof, one

- substitutes **Gaussian distributions/theta functions**, i.e., in essence, **exponentiations** of the **natural symplectic form** $\langle -, - \rangle_E$, for $\langle -, - \rangle_E$, and, moreover,
- allows for **arbitrary iterates** of the $\mathfrak{log}$-**link**, which, in effect, allow one to "*disguise*" the effects of such *exponentiation* operations,

then inter-universal Teichmüller theory bears **numerous striking resemblances** to Bogomolov's proof. Put another way, the **bridge** furnished by inter-universal Teichmüller theory between the *analogy* discussed in detail at the beginning of Sect. 5

(A1) between the *geometry surrounding* $E^{\angle}$ in Bogomolov's proof and the *combinatorics involving* ***l*-torsion points** that underlie the structure of $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters in inter-universal Teichmüller theory, on the one hand,

and the *analogy* discussed extensively in (L1–L7)

(A2) between the *geometry surrounding* $E^{\angle}$ in Bogomolov's proof and the **holomorphic/ mono-analytic log-shells**—i.e., in essence, the *local unit groups* associated to various completions of a number field—that occur in inter-universal Teichmüller theory, on the other

—i.e., the *bridge* between *l-torsion points* and *log-shells*—may be understood as consisting of the following *apparatus* of inter-universal Teichmüller theory:

(GE) ***l*-torsion points** [cf. (A1)] are, as discussed above, closely related to **exponents** of functions, such as **theta functions** or **algebraic rational functions** (cf. the discussion of [8, Remark 2.3.3, (vi), (vii), (viii)]; [8, Figs. 2.5, 2.6, 2.7]); such functions give rise, via the operation of **Galois evaluation** (cf. [8, Remark 2.3.3, (i), (ii), (iii)]), to **theta values** and elements of **number fields**, which one regards as acting on **log-shells** [cf. (A2)] that are constructed in a situation in which one considers **arbitrary iterates** of the $\mathfrak{log}$-**link** (cf. [8, Fig. I.6]).

In the context of the analogies (A1), (A2), it is also of interest to observe that the multiradial containers that are ultimately used in inter-universal Teichmüller theory (cf. [8, Fig. I.6]; [8, Theorem A]) consist of *processions of mono-analytic log-shells*, i.e., collections of mono-analytic log-shells whose **labels** essentially correspond to the **elements** of $|\mathbb{F}_l|$ [i.e., the quotient of the set $\mathbb{F}_l$ by the natural action of $\{\pm 1\}$]. This observation is especially of interest in light of the following aspects of inter-universal Teichmüller theory:

(P1) in inter-universal Teichmüller theory, the prime l is regarded as being *sufficiently large* that the finite field $\mathbb{F}_l$ serves as a **"good approximation"** for $\mathbb{Z}$ (cf. [6, Remark 6.12.3, (i)]);

(P2) at each *non-archimedean prime* at which the elliptic curve over a number field under consideration has *stable bad reduction*, the copy of "$\mathbb{Z}$" that is approximated by $\mathbb{F}_l$ may be naturally identified with the **value group** associated to the non-archimedean prime (cf. [7, Remark 4.7.3, (i)]);

(P3) at each *archimedean prime* of the number field over which the elliptic curve under consideration is defined, a mono-analytic log-shell essentially corresponds to a closed ball of **radius** π, centered at the origin in a Euclidean space of dimension two and subject to $\pm$-**indeterminacies** (cf. [8, Proposition 1.2, (vii)]; [8, Remark 1.2.2, (ii)]).

That is to say, if one thinks in terms of the *correspondences*

$$\textbf{mono-analytic log-shells} \quad \longleftrightarrow \quad E^{\angle}\ (\cong \mathbb{S}^1),$$
$$\textbf{procession labels} \in |\mathbb{F}_l|\ (\twoheadleftarrow \mathbb{F}_l \approx \mathbb{Z}) \quad \longleftrightarrow \quad \mathbb{Z}\cdot\pi \stackrel{\sim}{\rightarrow} \mathrm{Aut}\left((E^{\angle})^{\sim}/E^{|\angle|}\right),$$

then the collection of data constituted by a **"procession of mono-analytic log-shells"** is *substantially reminiscent* of the **objects** $(E^{\angle})^{\sim}$ $(\cong \mathbb{R})$, $\mathbb{R}_{|\pi|}$—i.e., in essence, copies of $\mathbb{R}$,

$\mathbb{R}_{\geq 0}$ that are **subject to** $\mathrm{Aut}_{\pi}(\mathbb{R})$**-,** $\mathrm{Aut}_{\pi}(\mathbb{R}_{\geq 0})$**-indeterminacies**—that play a central role in Bogomolov's proof.

Before concluding, we observe that, in the context of the above discussion of the technique of *Galois evaluation* [cf. (GE)], which plays an important role in inter-universal Teichmüller theory, it is also perhaps of interest to note the following further *correspondences* between the two theories:

(GE1) The *multiradiality* apparatus of inter-universal Teichmüller theory *depends*, in an *essential* way, on the *supplementary* ***geometric dimension*** constituted by the *"geometric containers"* (cf. [8, Remark 2.3.3, (i), (ii)]) furnished by *theta functions* and *algebraic rational functions*, which give rise, via *Galois evaluation*, to the *theta values* and elements of *number fields* that act directly on processions of mono-analytic log-shells. That is to say, this multiradiality apparatus would *collapse* if one attempted to work with these theta values and elements of number fields *directly*. This state of affairs is *substantially reminiscent* of the fact that, in Bogomolov's proof, it does *not suffice* to work directly with actions of (unipotent or toral/non-unipotent) elements of $SL(E)$ ($\cong SL_2(\mathbb{R})$) on $E^{\angle}$; that is to say, it is of essential importance that one work with **liftings** to $\widetilde{SL}(E)$ of these elements of $SL(E)$, i.e., to make use of the *supplementary geometric dimension* constituted by the bundle $\omega^{\times}_{\mathcal{M}} \to \mathcal{M}$.

(GE2) The fact that the theory of Galois evaluation surrounding **theta values** plays a *somewhat more central, prominent* role in inter-universal Teichmüller theory (cf. [7, §1, §2, §3]; [8, §2]) than the theory of Galois evaluation surrounding *number fields* is reminiscent of the fact that the original exposition of Bogomolov's proof in [1] essentially treats only the case of **genus zero**, i.e., in effect, only the *central*

Table 2 Contrasts between corresponding aspects of the two theories

Inter-universal Teichmüller Theory	Bogomolov's proof
Gaussians/theta functions play a central, motivating role	**Gaussians/theta functions entirely absent**
Weil pairing on *l-torsion points* defined over $\mathbb{F}_l$, $\nexists\, \mathbb{F}_l \hookrightarrow \mathbb{Q}$	**Natural symplectic form $\langle -, - \rangle_E$ defined over $\mathbb{R}$, $\exists\, \mathbb{R} \hookrightarrow \mathbb{C}$**
Subtle passage between **étale-like, Frobenius-like objects via absolute anabelian algorithms, Kummer theory/ cyclotomic rigidity algorithms**	*Confusion* between **étale-like, Frobenius-like objects via $\mathbb{R} \hookrightarrow \mathbb{C}$**
Geometry of $\Theta^{\pm \mathrm{ell}}$*NF*-Hodge theaters	**Symplectic geometry of classical upper half-plane**
Gaussians/theta functions, i.e., exponentiations of $\langle -, - \rangle_E$	**Natural symplectic form $\langle -, - \rangle_E$**
Arbitrary iterates of **log-link**	*Single application* of **logarithm, i.e., $(E^{\angle})^{\sim} \twoheadrightarrow E^{\angle}$**
$\Theta^{\times\mu}_{\mathrm{LGP}}$**-link relates *multiradial representation via mono-analytic log-shells* to conventional theory of arithmetic line bundles on number fields**	**Discriminant modular form "χ" relates *symplectic geometry* "$SL(E) \curvearrowright \omega^{\angle}_{\widetilde{\mathcal{M}}}$" to conventional algebraic theory of line bundles/divisors on $\mathcal{M}$**
Multiradial representation, inter-universality	*Non-holomorphic,* **real analytic nature of symplectic geometry**

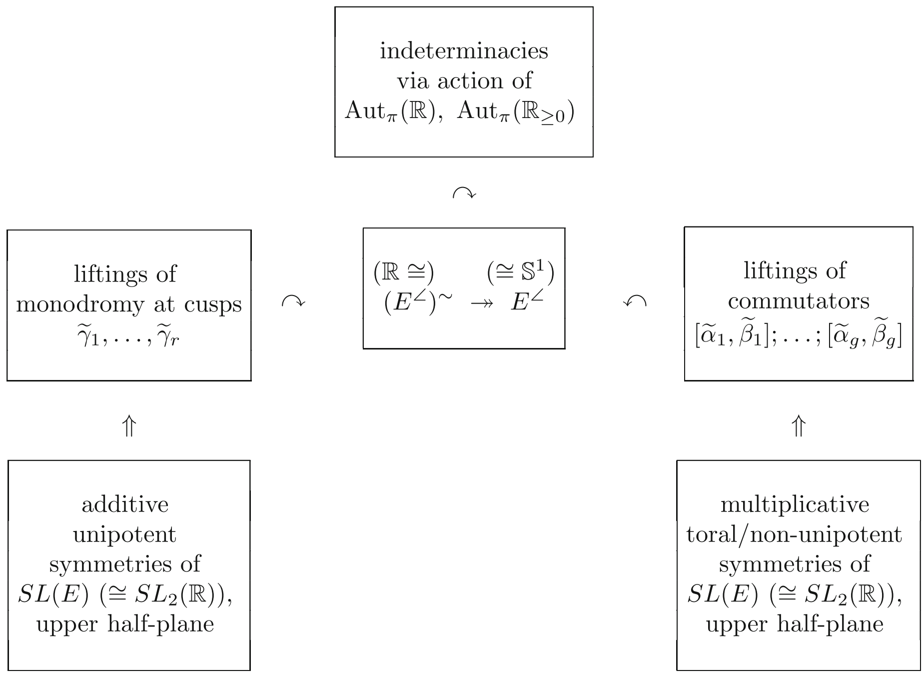

Fig. 5 The real analytic estimation apparatus of Bogomolov's proof

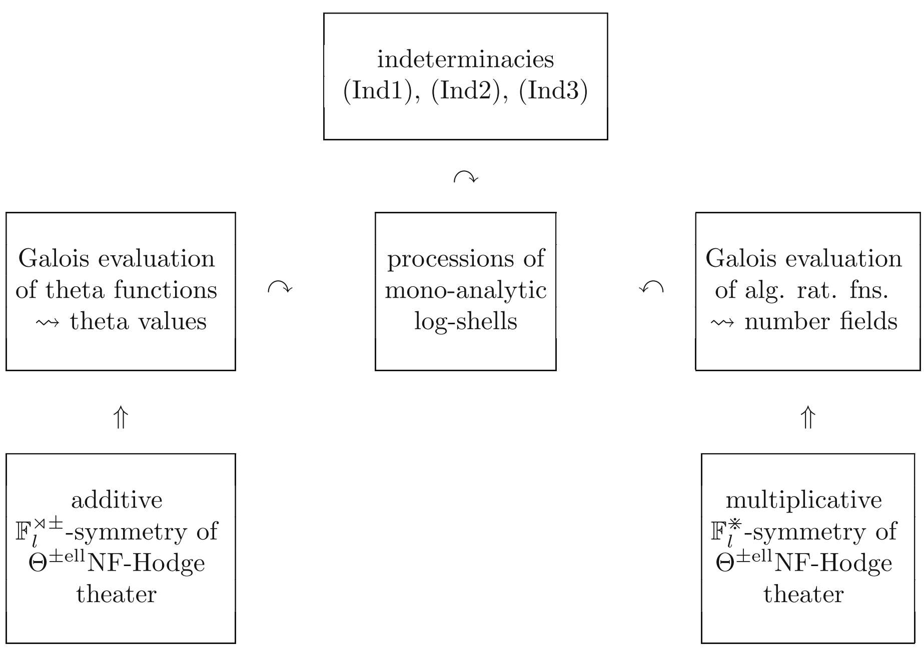

Fig. 6 The multiradial estimation apparatus of inter-universal Teichmüller theory

estimate of (B1), thus allowing one to ignore the *estimates concerning commutators* of (B2). It is only in the later exposition of [10] that one can find a detailed treatment of the estimates of (B2).

We conclude by observing that the numerous striking resemblances discussed above are perhaps *all the more striking* in light of the **complete independence** of the development of inter-universal Teichmüller theory from developments surrounding Bogomolov's proof: That is to say, the author was *completely ignorant* of Bogomolov's proof during the development of inter-universal Teichmüller theory. Moreover, inter-universal Teichmüller theory arose *not* as a result of efforts to "generalize Bogomolov's proof by substituting exponentiations of $\langle -, - \rangle_E$ for $\langle -, - \rangle_E$," but rather as a result of efforts (cf. the discussion of [2, §1.5.1, §2.1]; [4, Remarks 1.6.2, 1.6.3]) to overcome obstacles to applying scheme-theoretic Hodge–Arakelov theory to diophantine geometry by developing some sort of **arithmetic analogue** of the **classical functional equation** of the **theta function**. That is to say, despite the fact that the *starting point* of such efforts, namely the classical functional equation of the theta function, was **entirely absent** from the theory surrounding Bogomolov's proof, the theory, namely inter-universal Teichmüller theory, that ultimately arose from such efforts turned out, in hindsight, as discussed above, to be **remarkably similar** in numerous aspects to the theory surrounding Bogomolov's proof.

The content of the above discussion is summarized in Table 2. Also, certain aspects of our discussion—which, roughly speaking, concern the respective **"estimation apparatuses"** that occur in the two theories—are illustrated in Figs. 5 and 6. Here, we note that the mathematical content of Fig. 6 is essentially identical to the mathematical content of [8, Fig. I.6] (cf. also [6, Fig. I1.3]).

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