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Best rank- k approximations for tensors: generalizing Eckart–Young

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Abstract

Given a tensor f in a Euclidean tensor space, we are interested in the critical points of the distance function from f to the set of tensors of rank at most k , which we call the critical rank-at-most- k tensors for f . When f is a matrix, the critical rank-one matrices for f correspond to the singular pairs of f . The critical rank-one tensors for f lie in a linear subspace H_f , the critical space of f . Our main result is that, for any k , the critical rank-at-most- k tensors for a sufficiently general f also lie in the critical space H_f . This is the part of Eckart–Young Theorem that generalizes from matrices to tensors. Moreover, we show that when the tensor format satisfies the triangle inequalities, the critical space H_f is spanned by the complex critical rank-one tensors. Since f itself belongs to H_f , we deduce that also f itself is a linear combination of its critical rank-one tensors.

Keywords: Tensor, Eckart–Young Theorem, Best rank- k approximation

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1 Introduction

The celebrated Eckart–Young Theorem says that, for a real $m \times n$ -matrix A with $m \leq n$ and for an integer $k \leq m$, a matrix B of rank at most k nearest to A is obtained from A as follows: Compute the singular value decomposition $A = U\Sigma V^T$, where U, V are orthogonal matrices and where $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_m)$ is the “diagonal” $m \times n$ -matrix with the singular values $\sigma_1 \geq \dots \geq \sigma_m \geq 0$ on its main diagonal, and set $B := U\text{diag}(\sigma_1, \dots, \sigma_k, 0, \dots, 0)V^T$. Such a best rank- k approximation is unique if and only if $\sigma_k > \sigma_{k+1}$, and for us “nearest” refers to the Frobenius norm (but in fact, the result holds for arbitrary $O_m \times O_n$ -invariant norms [12]).

For higher-order tensors, an analogous approach for finding best rank- k approximations fails in general [18]. It succeeds, with respect to the Frobenius norm, for *orthogonally decomposable tensors* [1, 18], but this is a very low-dimensional real-algebraic variety in the space of all tensors. In this paper, we will establish versions of the Eckart–Young Theorem and the Spectral Theorem that *do* hold for general tensors.

To motivate this theorem, consider matrices once again, and assume that the σ_i are distinct and positive. A statement generalizing the Eckart–Young Theorem says that we obtain all critical points of the distance function $d_A(B) := \|A - B\|^2$ on the manifold of rank- k matrices by setting any $m - k$ of the singular values equal to zero [3], so as to obtain a matrix

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$$B_{i_1, \dots, i_k} := U \text{diag}(0, \dots, 0, \sigma_{i_1}, 0, \dots, 0, \sigma_{i_2}, 0, \dots, 0, \sigma_{i_k}, 0, \dots) V^T$$

for any ordered k -tuple $i_1 < \dots < i_k$ in $\{1, \dots, m\}$. We will call these critical points *critical rank- k matrices* for A . In particular, the critical rank-one matrices are B_1, \dots, B_m , and we draw attention to the fact that for each $k \geq 1$ and each k -tuple $i_1 < \dots < i_k$ the critical rank- k matrix B_{i_1, \dots, i_k} lies in the linear span of B_1, \dots, B_m . Moreover, this linear span has a direct description in terms of A : It consists of all matrices B such that both $A^T B$ and AB^T are symmetric matrices.

Taking cue from this observation, we will associate a *critical space* H_f to a tensor f show that H_f contains the critical rank-at-most- k tensors for f for each value of k (see below for a definition), and that H_f is spanned by the critical rank-one tensors for f . We will establish these results for sufficiently general partially symmetric tensors, and we work over the base field \mathbb{C} rather than \mathbb{R} .

Theorem 1.1 *Let f be a sufficiently general tensor in $S^{d_1} \mathbb{C}^{n_1+1} \otimes \dots \otimes S^{d_p} \mathbb{C}^{n_p+1}$. Then, for each natural number k , the critical rank-at-most- k tensors for f lie in the critical space H_f . Moreover, if for each ℓ with $d_\ell = 1$ the triangle inequality $n_\ell \leq \sum_{i \neq \ell} n_i$ holds, then $\text{codim } H_f = \sum_\ell \binom{n_\ell+1}{2}$ and H_f is spanned by the critical rank-one tensors for f . In particular, f itself lies in the linear span of the critical rank-one tensors for f .*

We record the following two corollaries over the real numbers.

Corollary 1.2 *If n_1, \dots, n_p satisfy the triangle inequality $n_\ell \leq \sum_{i \neq \ell} n_i$ for each $\ell = 1, \dots, p$, then for a sufficiently general tensor $f \in \otimes_{i=1}^p \mathbb{R}^{n_i+1}$ and any natural number k , any real tensor of real rank at most k closest to f in the Frobenius norm lies in the linear span of the complex critical rank-one tensors for f . In particular, f itself lies in the linear span of the complex critical rank-one tensors for f .*

Corollary 1.3 *For a sufficiently general symmetric tensor $f \in S^d \mathbb{R}^{n+1}$ and any natural number k , any real symmetric tensor of real symmetric rank at most k closest to f in the Frobenius norm lies in the linear span of the complex critical symmetric rank-one tensors for f . In particular, f itself lies in the linear span of the complex critical rank-one tensors for f .*

In the case of symmetric tensors, these critical rank-one tensors correspond to the so-called *eigenvectors* of f [11], while in the case of ordinary tensors, they correspond to *singular vector tuples* [10]. In the case $n = 1$ of binary forms, Corollary 1.3 was proved in [16]. The two corollaries above can be regarded as generalizations of the Eckart–Young Theorem and the Spectral Theorem from matrices to tensors.

Several remarks are in order here. First, we complexify d_f to the quadratic polynomial $d_f(u) := (u - f | u - f)$, where $(\cdot | \cdot)$ is the standard complex bilinear form on the space of tensors (and not a Hermitian form). The point of doing this is that, unlike for matrices, the critical points of this function on low-rank tensors are in general not real anymore, even if f is real. Accordingly, the critical space H_f , while defined by linear equations over \mathbb{R} if f is real, is taken to be the space of complex solutions to those equations. Second, we denote the dimensions by $n + 1$ rather than n since we will be using methods from projective algebraic geometry where the formulas look more appealing in terms of the projective dimension n than in the affine dimension $n + 1$. An example of this phenomenon is the

triangle inequalities in the theorem, which hold if and only if the variety dual to the Segre–Veronese embedding of the product $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_p}$ via degrees d_1, \dots, d_p is a hypersurface [7, Corollary 5.11].

The remainder of this paper is organized as follows. In Sect. 2, we define the critical space H_f for a partially symmetric tensor f and prove that the critical rank-at-most- k tensors for f lie in H_f . Then, in Sect. 3, we use vector bundle techniques to compute the dimension of the space spanned by the critical rank-one tensors for f and to show that this space equals H_f . In Sect. 4, we combine these ingredients to establish the results above.

2 The critical space of a tensor

2.1 Partially symmetric tensors and their ranks

Let $p \in \mathbb{Z}_{\geq 1}$, let V_1, \dots, V_p be complex vector spaces, and let $d_1, \dots, d_p \in \mathbb{Z}_{\geq 1}$. Let $S^d V$ be the d th symmetric power of V . We will study tensors in the space

$$T := S^{d_1} V_1 \otimes \dots \otimes S^{d_p} V_p.$$

So for $p = 1$, T is a symmetric power of V_1 , which is canonically isomorphic with the space of symmetric, d_p -way $n_1 \times \dots \times n_1$ -tensors. On the other hand, when all d_i are equal to 1, then T is a space of p -way ordinary tensors. We will write $[p] := \{1, \dots, p\}$.

Inside T , let X be the set of all tensors of the form

$$v_1^{d_1} \otimes \dots \otimes v_p^{d_p} \quad (v_\ell \in V_\ell, \ell \in [p]).$$

Then, X is a closed subvariety of T known as the affine cone over the Segre–Veronese embedding of $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_p}$ of degrees (d_1, \dots, d_p) . Let kX denote the set of sums of k elements of X . An arbitrary element t of T lies in kX for some k , and the minimal such k is called the rank of t [9, Definition 5.2.1.1]. For $p = 1$, this is the symmetric or Waring rank, and if all d_ℓ are 1, this notion is ordinary tensor rank. We write $\text{Sec}_k(X)$ for the Zariski (or Euclidean) closure of kX in T .

For real tensors, a few modifications are needed. The real rank of a real tensor t is the minimum k such that $t = \sum_{i=1}^k \lambda_i x_i$ with $\lambda_i \in \mathbb{R}$ and $x_i \in X_{\mathbb{R}}$ (it is enough to allow $\lambda_i = \pm 1$). For example $(e_1 + ie_2)^3 + (e_1 - ie_2)^3$ has complex rank 2 and real rank 3. Real rank is subtle for low-rank approximation of tensors. A classical example of de Silva and Lim [2] shows that for almost all $2 \times 2 \times 2$ -tensors of real rank 3 (like the above one) does not exist a closest tensor of real rank 2, while such phenomena may happen only for measure zero subsets of the set of complex tensors of given rank.

2.2 Symmetric bilinear forms and pairings

If V, W are complex vector spaces with symmetric bilinear forms (\cdot, \cdot) , and if $d \in \mathbb{Z}_{\geq 0}$, then $S^d V$ and $V \otimes W$ carry unique symmetric bilinear forms, also denoted (\cdot, \cdot) , that satisfy

$$\begin{aligned} (v^d | v'^d) &:= (v | v')^d \text{ and} \\ (v \otimes w | v' \otimes w') &:= (v | v')(w | w'). \end{aligned}$$

The first of these equalities implies

$$(v_1 \dots v_d | v'^d) = \prod_{i=1}^d (v_i | v')$$

and more in general

$$(v_1 \cdots v_d | v'_1 \cdots v'_d) = \frac{1}{d!} \sum_{\pi \in S_d} \prod_{i=1}^d (v_i | v'_{\pi(i)}).$$

We now fix nondegenerate symmetric bilinear forms on each V_ℓ , $\ell \in [p]$. Then, iterating these constructions, we obtain a canonical bilinear form on T .

Using the bilinear forms on V and W , we can also build more general bilinear maps whose outputs are vectors or tensors rather than scalars. We will call these bilinear maps *pairings* and denote them by $[\cdot, \cdot]$. Of particular relevance to us is the skew-symmetric pairing $S^d V \times S^d V \rightarrow \wedge^2 V$ determined by

$$[v^d | w^d] := (v | w)^{d-1} v \wedge w,$$

which implies

$$[v_1 \dots v_d | w^d] = \frac{1}{d} \sum_{i' \in [d]} \left(\prod_{i \neq i'} (v_i | w) \right) v_{i'} \wedge w$$

and more in general

$$[v_1 \cdots v_d | w_1 \cdots w_d] = \frac{1}{d \cdot d!} \sum_{i', j' \in [d]} \sum_{\pi: [d] \setminus i' \rightarrow [d] \setminus j'} \left(\prod_{i \neq i'} (v_i | w_{\pi(i)}) \right) v_{i'} \wedge w_{j'},$$

where π runs over all bijections $[d] \setminus i' \rightarrow [d] \setminus j'$.

Remark 2.1 In the case of binary forms ($\dim V = 2$ and arbitrary d), the pairing $[f | g]$ coincides (up to scalar multiples) with $(f | D(g))$, where $D(g) = g_{xy} - g_y x$; see [16]. Note the skew-symmetry property $(f | D(g)) = -(g | D(f))$. On the other hand, in the case of symmetric matrices ($d = 2$ and arbitrary V), the pairing $[f | g]$ coincides (up to scalar multiples) with the bracket $fg - gf$.

Building on this construction, for each $\ell \in [p]$ we define a pairing $[\cdot, \cdot]_\ell : T \times T \rightarrow \wedge^2 V_\ell$ by

$$[f_1 \otimes \cdots \otimes f_p | g_1 \otimes \cdots \otimes g_p]_\ell := \left(\prod_{i \neq \ell} (f_i | g_i) \right) [f_\ell | g_\ell], \quad f_i, g_i \in S^{d_i} V_i, \tag{1}$$

which we will use to define the critical space.

Remark 2.2 In the case of matrices $T = V_1 \otimes V_2$, the pairing $[f, g]_1$ coincides (up to scalar multiples) with $fg^t - g^t f$, while $[f, g]_2$ is (up to scalar multiples) $f^t g - g^t f$.

2.3 Basis, orthogonal basis and monomials

If B is a basis of V , then the degree- d monomials in the elements of B form a basis of $S^d V$. Such a basis is orthogonal if B is orthogonal. Hence, if we fix bases (respectively, orthogonal bases) of V_1, \dots, V_p , then by taking tensor products we obtain a basis (respectively, orthogonal basis) of T , whose elements we will call monomials of degree $D := \sum_{\ell=1}^p d_\ell$. We will use the word *gcd* of two such monomials x, y for the highest-degree monomial z such that both x and y can be obtained from z by multiplying z with suitable monomials.

Example 2.3 If $p = 3$ and $V_1 = V_2 = V_3 = \mathbb{C}^3$ with the standard bilinear form, and $d_1 = d_2 = 3$ and $d_3 = 2$, then the gcd of $(e_1^2 e_2) \otimes (e_1 e_2 e_3) \otimes (e_1^2)$ and $(e_1 e_2 e_3) \otimes (e_3^3) \otimes (e_2 e_3)$ equals $(e_1 e_2) \otimes (e_3) \otimes (1)$.

Lemma 2.4 For two monomials $f = f_1 \otimes \dots \otimes f_p, g = g_1 \otimes \dots \otimes g_p$ in T relative to the same orthogonal bases of V_1, \dots, V_p and for $\ell \in [p]$ we have $[f|g]_\ell = 0$ unless $f_i = g_i$ for all $i \neq \ell$ and $h := \gcd(f_\ell, g_\ell)$ has degree $d_\ell - 1$; in this case $\gcd(f, g)$ has degree $D - 1$ and $[u|v]_\ell \in \mathbb{C}^*(f_\ell/h) \wedge (g_\ell/h)$.

This is immediate from the definition of the pairing in (1).

2.4 Critical rank-one tensors

Let $f \in T$. Then, the critical points of the distance function $d_f : x \mapsto (f - x|f - x)$ on X are by definition those $x \in X \setminus \{0\}$ for which $f - x$ is perpendicular to the tangent space $T_x X$ to X at x ; we write this as $f - x \perp T_x X$. We call these tensors the *critical rank-one tensors* for f . For sufficiently general f , each of these critical rank-one tensors is *non-isotropic*, i.e., satisfies $(x|x) \neq 0$ (see [4, Lemma 4.2], in next Proposition 2.6 we will prove a slightly more general fact).

We will establish a bilinear characterization of these critical rank-one tensors for f . First, we describe the tangent space of X at a point x in more detail. For this, write

$$x = v_1^{d_1} \otimes \dots \otimes v_p^{d_p}. \tag{2}$$

Hence, we may extend each v_ℓ to a basis of V_ℓ . We then obtain an x -adapted basis of T consisting of monomials. If moreover x is non-isotropic, we have $(v_\ell|v_\ell) \neq 0$ and we may extend each v_ℓ to an orthogonal basis. We then obtain an x -adapted orthogonal basis of T .

Lemma 2.5 Let $x \in X$ as in (2).

- (1) Then, relative to any x -adapted basis, $T_x X$ is spanned by all degree- D monomials whose gcd with x has degree at least $D - 1$.
- (2) Assume moreover that x is non-isotropic. Then, relative to any x -adapted orthogonal basis, $(T_x X)^\perp$ is spanned by all degree- D monomials whose gcd with x has degree at most $D - 2$.

Proof Part (1) follows by applying the Leibniz rule to the parameterization (2) of X ; part (2) is a straightforward consequence. □

Proposition 2.6 Let $f \in T$ and let $x \in X$ be non-isotropic. Then, the following two statements are equivalent:

- (1) some (nonzero) scalar multiple of x is a critical rank-one tensor for f and
- (2) a unique (nonzero) scalar multiple of x is a critical rank-one tensor for f ;

and they imply the following statement:

- (3) for each $\ell \in [p]$, $[f|x]_\ell \in \wedge^2 V_\ell$ is zero.

Moreover, if f is sufficiently general, then every nonzero $x \in X$ satisfying (3) is non-isotropic and satisfies (1) and (2).

The pairing in item (3) is the pairing from (1).

Proof For the equivalence of the first two statements, we note that if $cx, c'x$ with $c, c' \neq 0$ are critical rank-one tensors for f , then $T_{cx}X = T_{c'x}X = T_xX$ and $f - cx \perp T_xX$ and $f - c'x \perp T_xX$. Since $x \in T_xX$, we find that $(c - c')x \perp x$, and using that x is non-isotropic we find that $c = c'$.

To prove that (1) implies (3), write x as in (2) and extend each v_ℓ to an orthogonal basis of V_ℓ , so as to obtain an x -adapted orthogonal basis of T . Now assume that $f - cx \perp T_xX$. Then, by Lemma 2.5, $f - cx$ is a linear combination of degree- D monomials whose gcds with x have degrees at most $D - 2$. Hence by Lemma 2.4, $[x|f - cx]_\ell = 0$. By the skew-symmetry, $[x|x]_\ell = 0$, so $[x|f]_\ell = 0$.

For the last statement, consider an $x = v_1^{d_1} \otimes \dots \otimes v_p^{d_p} \in X$ where, say, v_1, \dots, v_a with $a > 0$ are isotropic but the remaining factors are not. Extend each $v_\ell, \ell > a$ to an orthogonal basis of V_ℓ , and for v_ℓ with $\ell \leq a$ find an isotropic $w_\ell \in V_\ell$ with $(v_\ell|w_\ell) = 1$ and extend v_ℓ, w_ℓ with an orthogonal basis of the orthogonal complement of $\langle v_\ell, w_\ell \rangle^\perp$ to a basis of V_ℓ . In the corresponding (non-orthogonal) monomial basis of T , the monomials y with $[y|x]_\ell \neq 0$ for $\ell \leq a$ are those of the form

$$w_1^{d_1} \otimes \dots \otimes w_\ell^{d_\ell - 1} u_\ell \otimes \dots \otimes w_a^{d_a} \otimes v_{a+1}^{d_{a+1}} \otimes \dots \otimes v_p^{d_p},$$

where u_ℓ is a basis vector of V_ℓ that is distinct from v_ℓ but possibly equal to w_ℓ . These monomials all satisfy $[y|x]_i = 0$ for $i \neq \ell$. Similarly, the monomials y with $[y|x]_\ell \neq 0$ for $\ell > a$ are those of the form

$$w_1^{d_1} \otimes \dots \otimes w_a^{d_a} \otimes v_{a+1}^{d_{a+1}} \otimes \dots \otimes v_\ell^{d_\ell - 1} u_\ell \otimes \dots \otimes v_p^{d_p}$$

with u_ℓ a basis vector of V_ℓ distinct from v_ℓ ; they, too, satisfy $[y|x]_i = 0$ for $i \neq \ell$. The remaining monomials span the space of f s with $[x|f]_\ell = 0$ for all ℓ ; this space therefore has dimension

$$\dim T - (n_1 + \dots + n_p),$$

and it does not change when we scale x . Since the isotropic projective points $\langle x \rangle \in \mathbb{P}T$ form a subvariety of positive codimension in the $(n_1 + \dots + n_p)$ -dimensional projective variety $\mathbb{P}X$, the locus of all f for which there is a nonzero isotropic $x \in X$ with $[f|x]_\ell = 0$ for all ℓ has dimension less than $\dim T$.

Now assume that f is sufficiently general and let $x \in X \setminus \{0\}$ satisfy $[x|f]_\ell = 0$ for all ℓ . By the above, x is non-isotropic. Suppose that f , expanded on the x -adapted orthogonal basis, contains a monomial y whose gcd with x has degree exactly $D - 1$. If y agrees with x except in the factor $S^{d_\ell} V_\ell$ where it equals $v_\ell^{d_\ell - 1} u_\ell$, then in $[x|f]_\ell$, expanded on the standard basis of $\wedge^2 V_\ell$ relative to the chosen basis of V_ℓ , the term $v_\ell \wedge u_\ell$ has a nonzero coefficient. Hence, $[x|f]_\ell$ is nonzero, a contradiction.

Therefore, f contains only monomials whose gcds with x have degrees at most $D - 2$, and possibly the monomial x itself. Then, $f - cx \perp T_xX$ for a unique constant c . By generality of f , it does not lie in $(T_xX)^\perp$ for any $x \in X \setminus \{0\}$ (the union of these orthogonal complements is the cone over the variety dual to the projective variety defined by X , and of positive codimension). Hence, $c \neq 0$, and cx is a critical rank-one tensor for f . \square

Remark 2.7 The implication (1) \implies (3) in Proposition 2.6 holds without the assumption of non-isotropy of x . This follows from the fact that the ED correspondence

$$\{(x, f) \in X \times V \mid x \text{ is critical for } f\}$$

is a irreducible variety (see [3, §4 and Lemma 2.1]) and the nonempty open part in it where x is non-isotropic lies in the variety defined by $[f|x]_\ell = 0 \forall \ell \in [p]$ by Proposition 2.6.

2.5 The critical space

In view of Proposition 2.6, we introduce the following notion.

Definition 2.8 For a tensor $f \in T$, the critical space $H_f \subseteq T$ of f is defined as

$$H_f := \{g \in T \mid [f|g]_\ell = 0 \text{ for all } \ell \in [p]\}.$$

Remark 2.9 By the skew-symmetry, it follows immediately that $f \in H_f$.

Remark 2.10 In the case of binary forms ($\dim V = 2$), H_f is the hyperplane orthogonal to $D(f)$ [16]. In the case of ordinary tensors, H_f was first defined in [15] where it was called singular space, but in view of the results in this paper we feel that *critical space* is a better name.

Proposition 2.6 establishes that the non-isotropic critical rank-one tensors all lie inside H_f ; hence for a sufficiently general f , all critical rank-one tensors lie in H_f . In the next subsection, we will establish an analogous statement for higher ranks.

Note that the number of linear conditions for g to lie in H_f is at most $\sum_{\ell=1}^p \dim \wedge^2 V_\ell = \sum_{\ell=1}^p \binom{n_\ell+1}{2}$ —the linear conditions in the definition may not all be linearly independent. In Proposition 3.6 we will see that, assuming the triangle inequalities from Theorem 1.1 and assuming that f is sufficiently general, equality holds.

2.6 Higher rank

We will now establish a generalization of Proposition 2.6 to higher-rank tensors.

Definition 2.11 Let $f \in T$ and let k be any nonnegative integer. A critical rank-at-most- k tensor for f is a tensor $g \in kX$ such that $f - g \perp T_g \text{Sec}_k(X)$.

Note that by [4, Lemma 4.2], all the critical rank-at-most- k tensors for a sufficiently general $f \in T$ are smooth points of $\text{Sec}_k(X)$ and can be written as a sum of k non-isotropic rank-one tensors. Moreover, if we assume that k is at most the generic rank of tensors in T , then these critical tensors to a sufficiently general f have rank equal to k . If k is at least the generic rank of tensors in T , then the only critical rank-at-most- k tensor for a sufficiently general f is f itself.

Proposition 2.12 Let $f \in T$ be sufficiently general and let k be a nonnegative integer. Then, all the critical rank-at-most- k tensors for f lie in the critical space H_f .

Proof Let g be a critical rank-at-most- k tensor. By generality of f , g can be written as $x_1 + \dots + x_k$ with each $x_i \in X$ non-isotropic. Then, $T_g \text{Sec}_k X \supseteq \sum_{i=1}^k T_{x_i} X$, so that for each $i \in [k]$ we have $f - g \perp T_{x_i} X$. By Lemma 2.5 this means that, in the x_i -adapted orthogonal basis, $f - g$ is a linear combination of monomials whose gcds with x_i have degrees at most $D - 2$. Hence, by Lemma 2.4, $[f - g|x_i]_\ell = 0$ for all $\ell = 1, \dots, p$. We conclude that, for each ℓ ,

$$[f - g|g]_\ell = \sum_{i=1}^k [f - g|x_i]_\ell = 0,$$

and therefore

$$[f|g]_\ell = [f - g|g]_\ell + [g|g]_\ell = 0 + 0,$$

where in the last step we used that $[\cdot, \cdot]_\ell$ is skew-symmetric. Hence, $g \in H_f$. □

In the next section, we compute the dimension of the space spanned by the critical rank-one tensors for a general tensor and show that this space equals H_f .

3 The scheme of critical rank-one tensors

3.1 Critical rank-one tensors as the zero locus of a vector bundle section

Let $f \in T = \bigotimes_{\ell=1}^p S^{d_\ell} V_\ell$ be a tensor. We assume that $p \geq 2$, $d_\ell \geq 1$, and $\dim V_\ell = n_\ell + 1 \geq 1$ for all ℓ . We adapt the notation of [15, Section 5.1] to our current setting.

Consider the Segre–Veronese variety $\mathbb{P}X = \mathbb{P}V_1 \times \dots \times \mathbb{P}V_p$ embedded with $\mathcal{O}(d_1, \dots, d_p)$ in $\mathbb{P}T$; so $\mathbb{P}X$ is the projective variety associated with the affine cone $X \subseteq T$. Let $\pi_\ell : \mathbb{P}X \rightarrow \mathbb{P}V_\ell$ be the projection on the ℓ th factor and set $N := \dim \mathbb{P}X = n_1 + \dots + n_p$. For each $\ell \in [p]$ let \mathcal{Q}_ℓ be the quotient bundle on $\mathbb{P}V_\ell$, whose fiber over a point $\langle v \rangle$ is $V_\ell / \langle v \rangle$. From these quotient bundles, we construct the following vector bundles on $\mathbb{P}X$:

$$\mathcal{E} := \bigoplus_{\ell=1}^p \mathcal{E}_\ell \quad \text{where} \quad \mathcal{E}_\ell := (\pi_\ell^* \mathcal{Q}_\ell) \otimes \mathcal{O}(d_1, \dots, d_{\ell-1}, d_\ell - 1, d_{\ell+1}, \dots, d_p).$$

Note that \mathcal{E} has rank N . The fiber of \mathcal{E}_ℓ over a point $v := (\langle v_1 \rangle, \dots, \langle v_p \rangle) \in \mathbb{P}X$ consists of polynomial maps $\prod_{i=1}^p \langle v_i \rangle \rightarrow V_\ell / \langle v_\ell \rangle$ that are multi-homogeneous of multi-degree $(d_1, \dots, d_\ell - 1, \dots, d_p)$. The tensor f yields a global section of \mathcal{E}_ℓ which over the point v is the map sending $(c_1 v_1, \dots, c_p v_p)$ to the natural pairing of f with $(c_1 v_1)^{d_1} \dots (c_\ell v_\ell)^{d_\ell - 1} \dots (c_p v_p)^{d_p}$ —a vector in V_ℓ —taken modulo $\langle v_\ell \rangle$. Combining these p sections, f yields a global section s_f of \mathcal{E} . By Proposition 2.6, for f sufficiently general, the tensor $x := v_1^{d_1} \otimes \dots \otimes v_p^{d_p}$ is a nonzero scalar multiple of a critical rank-one tensor for f if and only if the point $(\langle v_1 \rangle, \dots, \langle v_p \rangle)$ is in the zero locus Z_f of the section s_f . In [5], this is used to compute the cardinality of Z_f for f sufficiently general as the top Chern class of \mathcal{E} . Our current task is different: we want to show that, if we assume the triangle inequalities of Theorem 1.1 and that f is sufficiently general, the linear span $\langle Z_f \rangle$ equals the projectivized critical space $\mathbb{P}H_f$; this is the second part of Theorem 1.1.

3.2 Bott’s formulas and a consequence

Our central tool will be the following formulas for the cohomology of vector bundles over projective spaces [13]. Recall that $\Omega_{\mathbb{P}^n}^r(k)$ is the $\mathcal{O}(k)$ -twisted sheaf of differential r -forms on \mathbb{P}^n .

Lemma 3.1 (Bott’s formulas) *For $q, n, r \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$, we have*

$$H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^r(k)) = \begin{cases} \binom{k+n-r}{k} \binom{k-1}{r} & \text{if } q = 0 \leq r \leq n \text{ and } k > r, \\ 1 & \text{if } 0 \leq q = r \leq n \text{ and } k = 0, \\ \binom{-k+r}{-k} \binom{-k-1}{n-r} & \text{if } q = n \geq r \geq 0 \text{ and } k < r - n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

A consequence featuring the triangle inequalities of Theorem 1.1 is the following.

Lemma 3.2 *Suppose that $n_\ell \leq \sum_{i \neq \ell} n_i$ holds for all ℓ with $d_\ell = 1$. Let $k \geq 2$ be an integer, q_1, \dots, q_p be nonnegative integers with $\sum_{\ell=1}^p q_\ell < k$ and r_1, \dots, r_p be nonnegative integers with $\sum_{\ell=1}^p r_\ell = k$. Then,*

$$\bigotimes_{\ell=1}^p H^{q_\ell} \left(\mathbb{P}V_\ell, \Omega_{\mathbb{P}V_\ell}^{r_\ell}(-d_\ell(k-1) + 2r_\ell) \right) = 0.$$

Proof Assume that all factors in the tensor product are nonzero.

First, if all of the factors were nonzero by virtue of the second and third line in Bott’s formulas, then we would have $q_\ell \geq r_\ell$ for all ℓ , and hence $k > \sum_\ell q_\ell \geq \sum_\ell r_\ell = k$, a contradiction.

So some factor is nonzero by virtue of the first line in Bott’s formulas; without loss of generality this is the first factor. Hence we have $q_1 = 0 \leq r_1 \leq n_1$ and $-d_1(k-1) + 2r_1 > r_1$. This last inequality reads $r_1 > d_1(k-1)$. Combining this with $\sum_\ell r_\ell = k$ and the fact that d_1 is a positive integer, we find that $r_1 = k, d_1 = 1$, and $r_\ell = 0$ for $\ell > 1$. In particular, there are no $\ell > 1$ for which the first line in Bott’s formulas applies.

For any $\ell > 1$, if the second line applies, then $0 = r_\ell = q_\ell = -d_\ell(k-1) + 2r_\ell$, which contradicts that both d_ℓ and $k-1$ are positive. Hence, the third line applies for all $\ell > 1$, and in particular we have $q_\ell = n_\ell$. But then

$$n_1 \geq r_1 = k > \sum_{l=1}^p q_l = \sum_{l=2}^p n_l,$$

which together with $d_1 = 1$ contradicts the triangle inequality in the lemma. □

3.3 Vanishing cohomology

The vanishing result in this subsection uses Lemma 3.2 and the following version of Künneth’s formula.

Lemma 3.3 *(Künneth’s formula) For vector bundles \mathcal{G}_ℓ on $\mathbb{P}V_\ell$ for $\ell = 1, \dots, p$ and a nonnegative integer q we have*

$$H^q \left(\mathbb{P}X, \bigotimes_{\ell} \pi_\ell^* \mathcal{G}_\ell \right) \cong \bigoplus_{q_1 + \dots + q_p = q} \bigotimes_{\ell} H^{q_\ell}(\mathbb{P}V_\ell, \mathcal{G}_\ell),$$

where the sum is over all p -tuples of nonnegative integers summing to q .

Lemma 3.4 *Suppose that $n_\ell \leq \sum_{i \neq \ell} n_i$ holds for all ℓ such that $d_\ell = 1$. Let $k \geq 2$ be an integer and $q \in \{0, \dots, k-1\}$. Then, we have*

$$H^q \left(\mathbb{P}X, \left(\bigwedge^k \mathcal{E}^* \right) \otimes \mathcal{O}(d_1, \dots, d_p) \right) = 0.$$

Proof First,

$$\mathcal{E}^* = \bigoplus_{\ell=1}^p (\pi_\ell^* \mathcal{Q}_\ell^*) \otimes \mathcal{O}(-d_1, \dots, -d_{\ell-1}, -(d_\ell - 1), -d_{\ell+1}, \dots, -d_p).$$

A well-known formula for k th wedge power of a direct sum yields

$$\bigwedge^k \mathcal{E}^* = \bigoplus_{r_1 + \dots + r_p = k} \bigotimes_{\ell} \bigwedge^{r_\ell} (\pi_\ell^* \mathcal{Q}_\ell^* \otimes \mathcal{O}(-d_1, \dots, -(d_\ell - 1), \dots, -d_p)).$$

Using $\bigwedge^r(\mathcal{F} \otimes \mathcal{O}(\omega)) = (\bigwedge^r \mathcal{F})(r\omega)$, $\mathcal{Q}^* = \Omega^1(1)$, and $\bigwedge^r(\Omega^1(1)) = \Omega^r(r)$, we obtain

$$\bigwedge^k \mathcal{E}^* = \bigoplus_{r_1+\dots+r_p=k} \bigotimes_{\ell} \left(\pi_{\ell}^* \Omega_{\mathbb{P}V_{\ell}}^{r_{\ell}}(r_{\ell}) \otimes \mathcal{O}(-r_{\ell}d_1, \dots, -r_{\ell}(d_{\ell}-1), \dots, -r_{\ell}d_p) \right).$$

Twisting by $\mathcal{O}(d_1, \dots, d_p)$, regrouping in each projection, and using $\sum_{\ell} r_{\ell} = k$ we find:

$$\left(\bigwedge^k \mathcal{E}^* \right) \otimes \mathcal{O}(d_1, \dots, d_p) = \bigoplus_{r_1+\dots+r_p=k} \bigotimes_{\ell} \pi_{\ell}^* \Omega_{\mathbb{P}V_{\ell}}^{r_{\ell}}(-d_{\ell}(k-1) + 2r_{\ell}).$$

To compute H^q of each summand we apply Künneth’s formula, and obtain subsummands which are exactly of the form in Lemma 3.2, hence zero. \square

3.4 Comparing $\mathbb{P}H_f$ and $\langle Z_f \rangle$

Assume that f is sufficiently general in T . By the first subsection of this section and by Proposition 2.6, Z_f is contained in the projectivized critical space $\mathbb{P}H_f$, hence so is $\langle Z_f \rangle$. Our goal now is to show that $\langle Z_f \rangle$ is equal to $\mathbb{P}H_f$ and to compute its dimension. Both of these goals are achieved through the following lemma. The section s_f of \mathcal{E} yields a homomorphism $\mathcal{E}^* \rightarrow \mathcal{O}$ of sheaves whose image is contained in the ideal sheaf \mathcal{I}_{Z_f} of the zero locus of s_f .

Lemma 3.5 *Assume that for each $\ell \in [p]$ we have $n_{\ell} \leq \sum_{i \neq \ell} n_i$ and that f is sufficiently general. Then, the induced homomorphism $\mathcal{E}^* \otimes \mathcal{O}(d_1, \dots, d_p) \rightarrow \mathcal{I}_{Z_f} \otimes \mathcal{O}(d_1, \dots, d_p)$ induces an isomorphism at the level of global sections.*

The following proof can be shortened considerably using spectral sequences, but we found it more informative in its current form. To make the formulas more transparent, we write $H^q(\cdot)$ instead of $H^q(\mathbb{P}X, \cdot)$ everywhere.

Proof To establish the desired isomorphism

$$H^0(\mathcal{E}^* \otimes \mathcal{O}(d_1, \dots, d_p)) \cong H^0(\mathcal{I}_{Z_f} \otimes \mathcal{O}(d_1, \dots, d_p))$$

we use the following Koszul complex (see, e.g., [8, Chapter III, Proposition 7.10A]):

$$0 = \bigwedge^{N+1} \mathcal{E}^* \rightarrow \bigwedge^N \mathcal{E}^* \rightarrow \dots \rightarrow \bigwedge^2 \mathcal{E}^* \rightarrow \mathcal{E}^* \rightarrow \mathcal{I}_Z \rightarrow 0.$$

Letting \mathcal{F}_k be the quotient of $\bigwedge^k \mathcal{E}^*$ by the image of $\bigwedge^{k+1} \mathcal{E}^*$, this yields the short exact sequence

$$0 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{E}^* \rightarrow \mathcal{I}_Z \rightarrow 0.$$

Tensoring with $\mathcal{O}(d_1, \dots, d_p)$ yields the short exact sequence

$$0 \rightarrow \mathcal{F}_2 \otimes \mathcal{O}(d_1, \dots, d_p) \rightarrow \mathcal{E}^* \otimes \mathcal{O}(d_1, \dots, d_p) \rightarrow \mathcal{I}_Z \otimes \mathcal{O}(d_1, \dots, d_p) \rightarrow 0,$$

and this gives a long exact sequence in cohomology beginning with

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{F}_2 \otimes \mathcal{O}(d_1, \dots, d_p)) &\rightarrow H^0(\mathcal{E}^* \otimes \mathcal{O}(d_1, \dots, d_p)) \rightarrow H^0(\mathcal{I}_Z \otimes \mathcal{O}(d_1, \dots, d_p)) \\ &\rightarrow H^1(\mathcal{F}_2 \otimes \mathcal{O}(d_1, \dots, d_p)) \rightarrow \end{aligned}$$

So to obtain the desired isomorphism we want that

$$H^q(\mathcal{F}_2 \otimes \mathcal{O}(d_1, \dots, d_p)) = 0 \quad \text{for } q = 0, 1.$$

For each $k = 2, \dots, N$, we have the short exact sequence

$$0 \rightarrow \mathcal{F}_{k+1} \rightarrow \bigwedge^k \mathcal{E}^* \rightarrow \mathcal{F}_k \rightarrow 0$$

which yields the long exact sequence

$$\begin{aligned} &\rightarrow H^{k-2} \left(\bigwedge^k \mathcal{E}^* \otimes \mathcal{O}(d_1, \dots, d_p) \right) \rightarrow H^{k-2} (\mathcal{F}_k \otimes \mathcal{O}(d_1, \dots, d_p)) \\ &\rightarrow H^{k-1} (\mathcal{F}_{k+1} \otimes \mathcal{O}(d_1, \dots, d_p)) \rightarrow H^{k-1} \left(\bigwedge^k \mathcal{E}^* \otimes \mathcal{O}(d_1, \dots, d_p) \right) \\ &\rightarrow H^{k-1} (\mathcal{F}_k \otimes \mathcal{O}(d_1, \dots, d_p)) \rightarrow H^k (\mathcal{F}_{k+1} \otimes \mathcal{O}(d_1, \dots, d_p)) \rightarrow \end{aligned}$$

Using Lemma 3.4, the two leftmost spaces are zero, so that

$$\begin{aligned} H^{k-2} (\mathcal{F}_k \otimes \mathcal{O}(d_1, \dots, d_p)) &\cong H^{k-1} (\mathcal{F}_{k+1} \otimes \mathcal{O}(d_1, \dots, d_p)) \text{ and} \\ H^{k-1} (\mathcal{F}_k \otimes \mathcal{O}(d_1, \dots, d_p)) &\subseteq H^k (\mathcal{F}_{k+1} \otimes \mathcal{O}(d_1, \dots, d_p)). \end{aligned}$$

Hence, using that $\mathcal{F}_{N+1} = 0$, we find that

$$\begin{aligned} H^0 (\mathcal{F}_2 \otimes \mathcal{O}(d_1, \dots, d_p)) &\cong \dots \cong H^{N-1} (\mathcal{F}_{N+1} \otimes \mathcal{O}(d_1, \dots, d_p)) = 0 \text{ and} \\ H^1 (\mathcal{F}_2 \otimes \mathcal{O}(d_1, \dots, d_p)) &\subseteq \dots \subseteq H^N (\mathcal{F}_{N+1} \otimes \mathcal{O}(d_1, \dots, d_p)) = 0, \end{aligned}$$

as desired. □

Proposition 3.6 *Suppose that for each $\ell \in [p]$ we have $n_\ell \leq \sum_{i \neq \ell} n_i$ and that f is sufficiently general. Then, $\langle Z_f \rangle = \mathbb{P}H_f$ and $\text{codim}_T H_f = \sum_\ell \binom{n_\ell+1}{2}$.*

Proof Since $\mathbb{P}X$ is embedded by $\mathcal{O}(d_1, \dots, d_p)$, the space of linear forms on T vanishing on Z_f is $H^0(\mathcal{I}_{Z_f} \otimes \mathcal{O}(d_1, \dots, d_p))$. By Lemma 3.5, this space is isomorphic to

$$\begin{aligned} H^0(\mathcal{E}^* \otimes \mathcal{O}(d_1, \dots, d_p)) &= \bigoplus_\ell H^0(\pi_\ell^* \mathcal{Q}_\ell^* \otimes \mathcal{O}(0, \dots, 1, \dots, 0)) \\ &= \bigoplus_\ell H^0(\pi_{\ell*}(\Omega_{\mathbb{P}V_\ell}^1(2))) = \bigoplus_\ell H^0(\mathbb{P}V_\ell, \Omega_{\mathbb{P}V_\ell}^1(2)), \end{aligned}$$

which by the first line in Bott’s formulas has dimension $\sum_\ell \binom{n_\ell+1}{2}$. This means that $\text{codim}_{\mathbb{P}T} \langle Z_f \rangle = \sum_\ell \binom{n_\ell+1}{2}$, so the second statement in the proposition follows from the first statement.

To establish the first statement, we spell out the map

$$H^0(\mathbb{P}V_\ell, \mathcal{Q}_\ell^* \otimes \mathcal{O}(1)) = H^0(\mathbb{P}V_\ell, \Omega_{\mathbb{P}V_\ell}^1(2)) \rightarrow H^0(\mathcal{I}_{Z_f} \otimes \mathcal{O}(d_1, \dots, d_p))$$

in greater detail. The space on the left is canonically $(\bigwedge^2 V_\ell)^*$, and an element ξ in this space is mapped to the linear form $T \rightarrow \mathbb{C}, g \mapsto \xi([f|g]_\ell)$. As ℓ varies, these are precisely the linear forms that cut out H_f . This proves that $\mathbb{P}H_f = \langle Z_f \rangle$. □

Remark 3.7 In general, for the equality $\langle Z_f \rangle = \mathbb{P}H_f$ we only need that the linear equations cutting out $\mathbb{P}H_f$ also cut out Z_f , i.e., we only need that the linear map in Lemma 3.5 is surjective. One might wonder whether this surjectivity remains true when the triangle inequalities fail. In the case of $(n_1 + 1) \times (n_2 + 1)$ -matrices, it does indeed—there we already knew the critical rank-one approximations span the critical space—but for $p = 3$ and $2 \times 2 \times 4$ -tensors (so that $n_3 = 3 > 1 + 1 = n_1 + n_2$) the space $\langle Z_f \rangle$ has dimension 6 while computer experiments suggest that the space $\mathbb{P}H_f$ has dimension 7, hence the surjectivity fails. Still, in these experiments, f itself seems to lie in the span of Z_f . This leads to the open problem whether our analogue of the Spectral Theorem and the Eckart–Young Theorem persists when the triangle inequalities fail.

4 Proofs of the main results

Proof of Theorem 1.1 The first statement is Proposition 2.12; the second and third statement are Proposition 3.6. The last statement follows from Remark 2.9. \square

Proof of Corollaries 1.2 and 1.3. If g is a real tensor of real rank at most k closest to f , then one can write it as $x_1 + \dots + x_k$ with x_1, \dots, x_k real points of X . In particular, all of these points are non-isotropic, and the argument of Proposition 2.12 applies. Hence, g lies in H_f . Now the result follows from Proposition 3.6. The argument applies, in particular, to k equal to the rank of f , which gives the last statement of the corollaries. \square

Note that, if f is *any* real tensor, then any real tensor of real rank at most k closest to f lies in H_f by the argument above. Only for the conclusion that it lies in the span of the complex critical rank-one tensors of f do we use that f is sufficiently general. We do not know whether this generality is really needed.

Also note that we do not shed new light on the question of when for sufficiently general f there *exists* a closest real tensor of rank at most k . For an update on the complex case, see [17].

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References

- Boralevi, A., Draisma, J., Horobej, E., Robeva, E.: Orthogonal and unitary tensor decomposition from an algebraic perspective. *Isr. J. Math.* **222**, 223–260 (2017)
- de Silva, V., Lim, L.-H.: Tensor rank and the ill-posedness of the best low-rank approximation problem. *SIAM J. Matrix Anal. Appl.* **30**, 1084–1127 (2008)
- Draisma, J., Horobej, E., Ottaviani, G., Sturmfels, B., Thomas, R.: The Euclidean distance degree of an algebraic variety. *Found. Comput. Math.* **16**(1), 99–149 (2016)
- Drusvyatskiy, D., Lee, H.-L., Ottaviani, G., Thomas, R.: The Euclidean distance degree of orthogonally invariant matrix varieties. *Isr. J. Math.* **221**, 291–316 (2017)
- Friedland, S., Ottaviani, G.: The number of singular vector tuples and uniqueness of best rank one approximation of tensors. *Found. Comput. Math.* **14**(6), 1209–1242 (2014)
- Friedland, S., Tammali, V.: Low-rank approximation of tensors. In: Benner, P., Bollhöfer, M., Kressner, D., Mehl, C., Stykel, T. (eds.) *Numerical Algebra, Matrix Theory, Differential-Algebraic Equations and Control Theory*. Springer, Cham (2015)
- Gelfand, I., Kapranov, M., Zelevinsky, A.: *Discriminants, Resultants and Multidimensional Determinants*. Birkhäuser, Boston (1994)
- Hartshorne, R.: *Algebraic Geometry*, Graduate Texts in Mathematics 52. Springer, New York (1977)
- Landsberg, J.M.: *Tensors: Geometry and Applications*, Volume 128 of Graduate Studies in Mathematics. American Mathematical Society (AMS), Providence (2012)

10. Lim, L.H.: Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of IEEE International Workshop on Computing Advances in Multi-sensor Adaptive Processing (CAMSAP 2005), pp. 129–132
11. Maccioni, M.: The number of real eigenvectors of a real polynomial, to appear in *Bollettino dell'Unione Matematica Italiana*. [arXiv:1606.04737](https://arxiv.org/abs/1606.04737)
12. Mirsky, L.: Symmetric gauge functions and unitarily invariant norms. *Q. J. Math. Oxf. II Ser.* **11**, 50–59 (1960)
13. Okonek, C., Schneider, M., Spindler, H.: *Vector Bundles on Complex Projective Spaces*, Progress in Mathematics, vol. 3. Birkhäuser, Boston (1980)
14. Oeding, L., Ottaviani, G.: Eigenvectors of tensors and algorithms for waring decomposition. *J. Symb. Comput.* **54**, 9–35 (2013)
15. Ottaviani, G., Paoletti, R.: A geometric perspective on the singular value decomposition. *Rend. Istit. Mat. Univ. Trieste* **47**, 107–125 (2015)
16. Ottaviani, G., Tocino, A.: Best rank k approximation for binary forms. *Collect. Math.* **69**, 163–171 (2018)
17. Qi, Y., Michałek, M., Lim, L.H.: Complex tensors almost always have best low-rank approximations. preprint. [arXiv:1711.11269](https://arxiv.org/abs/1711.11269)
18. Vannieuwenhoven, N., Nicaise, J., Vandebril, R., Meerbergen, K.: On generic nonexistence of the Schmidt–Eckart–Young decomposition for complex tensors. *SIAM J. Matrix Anal. Appl.* **35**(3), 886–903 (2014)